

Novel Min-Max Reformulations of Linear Inverse Problems

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Abstract

In this article, we dwell into the class of so-called ill-posed Linear Inverse Problems (LIP) which simply refer to the task of recovering the entire signal from its relatively few random linear measurements. Such problems arise in a variety of settings with applications ranging from medical image processing, recommender systems, etc. We propose a slightly generalized version of the error constrained linear inverse problem and obtain a novel and equivalent convex-concave min-max reformulation by providing an exposition to its convex geometry. Saddle points of the min-max problem are completely characterized in terms of a solution to the LIP, and vice versa. Applying simple saddle point seeking ascend-descent type algorithms to solve the min-max problems provides novel and simple algorithms to find a solution to the LIP. Moreover, the reformulation of an LIP as the min-max problem provided in this article is crucial in developing methods to solve the dictionary learning problem with almost sure recovery constraints.

Keywords: linear inverse problems, min-max problems, dictionary learning.

1. Introduction

A Linear Inverse Problem (LIP) is, simply stated, the recovery of a signal from its linear measurements. Signals encountered in practise tend to be very high dimensional; for example, audio signals and images typically have ambient dimension ranging from a few thousands to millions. However, the number of linear measurements that are typically available to recover the entire signal from, are relatively few compared to their ambient dimension. This makes such an LIP ill-posed. Fortunately, high dimensional data of the present day and age often contain underlying low dimensional characteristics, which if taken into consideration, often suffice to overcome the ill-posedness of the problem.

One of the early instances that gave recognition to linear inverse problems is *compressed sensing* (Donoho, 2006; Candès and Wakin, 2008; Candès et al., 2006a,b), where a given signal f' is assumed to be sparse in some known basis. So, given the partial information of the signal in the form of a collection of linear measurements $x = \phi(f')$, the objective is to recover the entire signal almost accurately. Since the given signal is known to be sparse,

one would expect that the true signal can be recovered accurately by finding a sparsest solution to the under determined system of linear equations $x = \phi(f)$ given by the linear measurements. However, finding the sparsest solutions (i.e., having minimum ℓ_0 “norm”) to linear equations is NP-hard (Natarajan, 1995; Tropp, 2006), and therefore, not practical in the intended applications due to the size of the data typically encountered there. Fortunately, it is now well established that under mild conditions (which hold true in relevant problems), the simple convex heuristic of minimizing the ℓ_1 -norm

$$\begin{cases} \text{minimize} & \|f\|_1 \\ \text{subject to} & x = \phi(f), \end{cases} \quad (1)$$

instead of the ℓ_0 -penalty finds the true solution. Thus, the true signal can be recovered exactly by simply solving a convex optimization problem. Moreover, even if the linear measurements are noisy, recovery done via minimizing the ℓ_1 -penalty is reasonably accurate.

Similar to compressed sensing is the problem of low rank matrix recovery or completion (Candès and Recht, 2009; Recht et al., 2010; Chandrasekaran et al., 2012), where the objective is to reconstruct an entire matrix M' from only a few of its entries $[M']_{ij}$ for $(i, j) \in I$, where the cardinality of I is “small” compared to the size of the matrix M' . Matrix recovery or completion problems arise regularly in recommender systems, and the Netflix challenge case in point. Since the unknown matrix is known to be of low rank, one expects that the true matrix is the solution to the rank minimization problem:

$$\begin{cases} \text{minimize} & \text{rank}(M) \\ \text{subject to} & [M']_{ij} = [M]_{ij} \quad \text{for } (i, j) \in I. \end{cases}$$

However, minimizing the rank exactly, is extremely hard and impractical for most applications. Analogous to the ℓ_1 -minimization, it is now well established (Recht et al., 2010) that under mild conditions, the simple convex heuristic of minimizing the matrix nuclear norm $\|\cdot\|_*$, recovers the true low rank matrix.

$$\begin{cases} \text{minimize} & \|M\|_* \\ \text{subject to} & [M']_{ij} = [M]_{ij} \quad \text{for } (i, j) \in I. \end{cases} \quad (2)$$

Often, signals that are encountered in practice can be written as a linear combination of relatively few elements from some atomic set \mathcal{A} which depends on the low dimensional characteristics present in the signal. For instance, in compressed sensing, since the signal is assumed to be sparse, the atomic set \mathcal{A} is considered to be the standard Euclidean basis of appropriate dimension. In the matrix recovery problem, since the unknown matrix is assumed to be of low rank, it can be written as the sum of a few rank-1 matrices, and thus the atomic set \mathcal{A} is the set of all rank-1 matrices of appropriate dimensions. So, given such a signal with the corresponding atomic set \mathcal{A} , an LIP attempts to find a linear combination of few elements from the atomic set \mathcal{A} that agree with the given linear measurements of the signal. However, as evident in the compressed sensing and matrix recovery problems, finding such a linear combination by simply searching the atomic set is impractical.

It is to be observed that the ℓ_1 and the nuclear norms are the *guage* functions corresponding to the convex hull of the standard Euclidean basis (atomic set in compressed sensing) and the set of rank-1 matrices (atomic set in matrix recovery problem) respectively. By minimizing such convex functions subject to the linear measurement constraints, guarantees have been obtained for fruitful signal recovery in compressed sensing and matrix recovery problems. Motivated by this observation, it was established in (Chandrasekaran et al., 2012) that for a generic LIP with a generic atomic set \mathcal{A} , the analogous convex heuristic of minimizing the guage function corresponding to the set $\text{conv}(\mathcal{A})$ provides exact recovery under mild conditions. Thus, given linear measurements $x = \phi(f')$ of a signal f' , an LIP seeks to solve

$$\begin{cases} \underset{f}{\text{minimize}} & c(f) \\ \text{subject to} & x = \phi(f), \end{cases} \quad (3)$$

where c is a positively homogenous convex cost function such that $\text{conv}(\mathcal{A}) = \{f : c(f) \leq 1\}$. If the observed linear measurements are noisy, i.e., $x = \phi(f) + \xi$, for some measurement noise ξ . We solve

$$\begin{cases} \underset{f}{\text{minimize}} & c(f) \\ \text{subject to} & \|x - \phi(f)\| \leq \epsilon, \end{cases} \quad (4)$$

where $\epsilon \geq 0$ is chosen based on the statistical properties of the measurement noise ξ . If the true signal f' is a linear combination of only a ‘‘few’’ elements of \mathcal{A} , it can be recovered from its linear measurements by simply solving the LIP (4). A great body of literature (Donoho, 2006; Candès and Wakin, 2008; Candès et al., 2006a,b; Candès and Recht, 2009; Recht et al., 2010; Chandrasekaran et al., 2012) exists on linear inverse problems focusing primarily towards providing quantitative analysis of the number of measurements required, and the constraints on the type of measurements suited for a given atomic set in order to guarantee fruitful recovery. However, our objective in studying the linear inverse problems is not directed towards this cause.

The main motivation for our work in this article comes from the related problem of *Dictionary Learning*, which is another well known machine learning problem. The objective in dictionary learning is to find a standard database of vectors called the *dictionary* such that samples of the given data $(x_t)_t$ can be expressed as linear combinations of the dictionary vectors with desirable features, an important one being sparsity. Due to the many benefits of sparse representation in applications such as compression, robustness, clustering etc., there is an ever increasing demand to learn good dictionaries that offer maximally sparse but also reasonably accurate representation of the data. A brief overview on the relevance of the dictionary learning problem and methods used to learn a ‘good’ dictionary are given in (Tosic and Frossard, 2011).

To this end, let $D = (d_1 \ d_2 \ \cdots \ d_K)$ denote a dictionary of K vectors, where K is some positive integer. Let f_t denote the representation of sample x_t for every $t \in 1, 2, \dots, T$.

Then the dictionary learning problem that we aim to solve is

$$\left\{ \begin{array}{l} \underset{(f_t)_t, D}{\text{minimize}} \quad \frac{1}{T} \sum_{t=1}^T \|f_t\|_1 \\ \text{subject to} \quad \left\{ \begin{array}{l} D \in \mathcal{D}, \\ \|x_t - Df_t\|_2 \leq \epsilon_t \quad \text{for every } t = 1, 2, \dots, T, \end{array} \right. \end{array} \right. \quad (5)$$

where \mathcal{D} is some convex subset of $\mathbb{R}^{n \times K}$ and $(\epsilon_t)_t$ is a given sequence of non-negative real numbers. In image processing applications like denoising etc., ϵ_t corresponds to the bounds on the additive noise in the noisy data. It is to be noted that for a fixed dictionary D , the optimization over $(f_t)_t$ simply consists of solving the LIP (4) for each t . Minimization of the ℓ_1 penalty is to enforce sparsity in the representation vectors $(f_t)_t$. For different applications, the dictionary can be learned to optimize a task-specific but otherwise generic cost function $c(\cdot)$ instead of the ℓ_1 -norm.

Conventionally, the dictionary learning problem is addressed by solving the optimization problem

$$\underset{(f_t)_t, D \in \mathcal{D}}{\text{minimize}} \quad \frac{1}{T} \sum_{t=1}^T \left(\|f_t\|_1 + \gamma \|x_t - Df_t\|_2^2 \right) \quad (6)$$

where $\gamma > 0$ is the regularization parameter. It should be noted that the objective function in (6) is a weighted sum of the sparsity inducing ℓ_1 -penalty $\|f_t\|_1$ and the error term $\|x_t - Df_t\|_2^2$. The regularization parameter γ influences the tradeoff between the level of sparsity and the error, and for a given value of γ , this tradeoff is specific to a given distribution or data set. However, the precise relation between the value of regularization parameter γ and the tradeoff is not straightforward. Thus, a priori one does not know which value of the regularization parameter γ to pick for a given distribution or data set; it is a tuning parameter that needs to be learned from the data.

The dictionary learning problem (5) differs from the mainstream one (6) in that it imposes constraint on every sample to be reconstruction within limits. Such a formulation arises naturally in many image processing applications like compressed sensing (Donoho, 2006; Candès and Wakin, 2008), inpainting, denoising problems in image processing (Elad and Aharon, 2006) etc., where, good estimates of $(\epsilon_t)_t$ to be used in (5) are known a priori. In contrast, if we were to learn the dictionary for the same applications but by solving (6) instead, the appropriate value of the regularization parameter to be used is not known and needs to be learnt from other techniques like cross validation, which poses additional computational challenges. Furthermore, with a single parameter the problem formulation (6) does not provide the level of customisability that is available in (5). Therefore, learning dictionaries by solving (5) is more appealing in situations where good estimates of $(\epsilon_t)_t$ to be used are known beforehand or in situations where the user has the liberty of specifying it.

Most of the existing techniques (Mairal et al., 2010; Aharon and Elad, 2006) that learn a dictionary by solving (6), do so by alternating the minimization over the variables $(f_t)_t$

and D keeping the other one fixed, i.e., alternating between the problems

$$\left\{ \begin{array}{l} \underset{(f_t)_t}{\text{minimize}} \quad \frac{1}{T} \sum_{t=1}^T \left(\|f_t\|_1 + \gamma \|x_t - Df_t\|_2^2 \right), \text{ and} \\ \underset{D \in \mathcal{D}}{\text{minimize}} \quad \frac{1}{T} \sum_{t=1}^T \|x_t - Df_t\|_2^2. \end{array} \right. \quad (7)$$

We note that, individually both the problems in (7) are convex, in particular, the optimization over the dictionaries is a *Quadratic Program (QP)*.

In contrast to (6), an alternating minimization technique like (7) applied directly to (5) is completely ineffective. Indeed, once the variables $(f_t)_t$ are fixed, there is no evident way to update the dictionary variable such that the resulting dictionary minimizes the cost. This makes the dictionary learning problem (5) ill-posed and more challenging than the conventional regularised formulation. We observe that the objective function in (6) depends directly on the dictionary variable D , whereas, it affects the objective function of (5) indirectly by only changing the set of feasible representations. This is the key reason which makes updating the dictionary in (5) such a difficult task.

In this article, we propose a slight modification to the original linear inverse problem (4), involving an additional parameter, which adds robustness to (4) when positive. Convex duality of the proposed problem is studied by exposing the underlying geometry. We propose convex-concave min-max problems and establish their equivalence to the modified LIP. Mathematical guarantees based on Fenchel duality relating the optimal solution of the LIP with the saddle points of the min-max problems are provided. This equivalent reformulation of the LIP as a min-max problem precisely addresses the issue of ill-posedness in the dictionary learning problem (5). In the dictionary learning problem (5), replacing the optimization over variables $(f_t)_t$, which is a collection of linear inverse problems, with their respective min-max problems, gives us another min-max problem equivalent to (5), but well-posed. Making use of this reformulation, we have provided novel dictionary learning algorithm in (Sheriff and Chatterjee, 2020) to solve (5). To the best of our knowledge, these are the first set of results that effectively solve the dictionary learning problem for situations where solving the formulation (5) is natural.

In addition to the importance of the reformulations of the LIP in dictionary learning, the min-max forms also provide a new approach to solve an LIP, which is of independent interest. Gradient based algorithms to compute saddle points of the min-max problem give rise to simple and easy to implement algorithms to solve an LIP. Due to the relevance of large dimensional linear inverse problems, simple to implement yet reasonably fast and efficient algorithms to solve them are always desirable. Furthermore, the theory and mathematical guarantees provided in this article are fairly generic and easily carry forward to the extended class of optimization problems which optimize a convex gauge function over convex sets, this includes many relevant problems like projections onto convex sets, LASSO etc., thereby yielding new algorithms for all such problems at once.

The article unfolds as follows: In Section 2 we formally introduce the LIP in a more generalised form and provide the main results including the equivalent convex-concave min-

max reformulations, algorithms to solve them, and applications to some specific problems of interest. In Section 3, we expose the duality by studying the underlying convex geometry of the LIP and provide proofs for all the results. We employ standard notations, and specific ones are explained as they appear.

2. Formal Problem Statement and Main Results

Let n be a positive integer, \mathbb{H}_n be an n -dimensional Hilbert space equipped with an inner-product $\langle \cdot, \cdot \rangle$ and its associated norm $\|\cdot\|$. For every $z \in \mathbb{H}_n$ and $r > 0$, let $B(z, r) := \{y \in \mathbb{H}_n : \|x - y\| < r\}$ and let $B[z, r] := \{y \in \mathbb{H}_n : \|x - y\| \leq r\}$. Let $c : \mathbb{R}^K \rightarrow [0, +\infty[$ be a cost function such that it satisfies the following assumption.

Assumption 1 *The cost function $c : \mathbb{R}^K \rightarrow [0, +\infty[$ has the following properties*

- *Positive Homogeneity : There exists a positive real number p such that for every $\alpha \geq 0$ and $f \in \mathbb{R}^K$, we have $c(\alpha f) = \alpha^p c(f)$.*
- *Pseudo-Convexity : The unit sublevel set $V_c := \{f \in \mathbb{R}^K : c(f) \leq 1\}$ is convex.*
- *Inf-Compactness : The unit sublevel set V_c is compact.*

Assumption 1 is enforced throughout the article for the cost function $c(\cdot)$ unless and otherwise stated explicitly. In addition to Assumption 1, if the cost function $c(\cdot)$ satisfies $c(-f) = c(f)$ and $p = 1$, then $c(\cdot)$ is a *norm*. In general, a cost function $c(\cdot)$ satisfying Assumption 1 and $p = 1$ falls under the class of *Minkowski gauge-functions* (Boyd and Vandenberghe, 2004, Exercise 3.34) which is a class of positively homogeneous convex functions.

Let $x \in \mathbb{H}_n$, non-negative real numbers ϵ and δ , and the linear map $\phi : \mathbb{R}^K \rightarrow \mathbb{H}_n$ be given. We consider the following general formulation of the linear inverse problem

$$\begin{cases} \text{minimize} & \mathbf{c}^p \\ (\mathbf{c}, f) \in \mathbb{R} \times \mathbb{R}^K & \\ \text{subject to} & \begin{cases} (c(f))^{1/p} \leq \mathbf{c} \\ \|x - \phi(f)\| \leq \epsilon + \delta \mathbf{c}. \end{cases} \end{cases} \quad (8)$$

When $\delta = 0$, we see that the second constraint is independent from the variable \mathbf{c} . Consequently, for every $f \in \mathbb{R}^K$ such that $\|x - \phi(f)\| \leq \epsilon$, the minimization over the variable \mathbf{c} is achieved for $\mathbf{c} = (c(f))^{1/p}$. Thus, the linear inverse problem (8) reduces to the following more familiar formulation.

$$\begin{cases} \text{minimize} & c(f) \\ f \in \mathbb{R}^K & \\ \text{subject to} & \|x - \phi(f)\| \leq \epsilon. \end{cases} \quad (9)$$

If $\delta = 0$, it is immediately apparent that (8) is feasible if and only if $B[x, \epsilon] \cap \text{image}(\phi) \neq \emptyset$. On the contrary, the linear inverse problem (8) becomes strictly feasible if $\delta > 0$ irrespective of x and ϕ . This is easily seen by considering $f = 0$ and $\mathbf{c} > (\|x\|/\delta)$ in (8).¹ Therefore,

1. We say that a convex problem is *strictly feasible* if and only if there exists a feasible point that satisfies all the inequalities strictly.

non-negative real number δ acts as a robustness parameter, particularly with regards to the feasibility of (8).

It might be surprising at first to see the rather unusual formulation (8) of the linear inverse problem. Our formulation makes way for the possibility of δ to take positive values also. By considering $\delta > 0$, we obtain several advantages:

- Whenever $\delta > 0$, the LIP (8) is always strictly feasible. This is a crucial feature in the initial stages of dictionary learning, in particular, when the data lies in a subspace of lower dimension m , such that $m, K \ll n$, where K is the number of dictionary vectors.
- A positive value of δ provides desirable properties like continuity, well defined gradients of the objective function in terms of the dictionary variable. Thus, having $\delta > 0$ imparts regularity to the dictionary learning problem (for e.g., well defined gradients with respect to the dictionary variable). Such properties in practice lead to well conditioning and numerical stability in algorithms.
- Having a positive value of δ provides guarantees for convergence of dictionary learning algorithms. Moreover, it leads to useful fixed point characterization of the optimal dictionary, which can be used to employ a fixed point seeking online dictionary update algorithms (Sheriff and Chatterjee, 2020, Algorithm 4).

We observe that the mapping $f \mapsto (c(f))^{1/p}$ is an inf-compact, convex and positively homogeneous of order 1. Therefore, it is immediate that the constraints of the LIP (8) are convex. Furthermore, the objective function is also convex whenever $p \geq 1$. Thus, it is apparent that the LIP (8) is a convex problem when $p \geq 1$. When $p < 1$, we highlight that $[0, +\infty[\ni (\cdot) \mapsto (\cdot)^p$ is an increasing function, and therefore, minimizing c^p is equivalent to minimizing c . Thus, the LIP (8) has an underlying convex problem (except that the objective function is a non-convex power).

We emphasise that whenever the optimization problem (8) is feasible, the collection of feasible (c, f) is a closed subset of $[0, +\infty[\times \mathbb{R}^K$. Moreover, the objective function c^p is continuous and *coercive* on the feasible set.² Therefore, from the Weierstrass theorem (Rudin, 1964, Theorem 4.16) we conclude that whenever (8) is feasible, it admits an optimal solution. To this end, let

$$((C_\delta(\phi, x, \epsilon))^{\frac{1}{p}}, F_\delta(\phi, x, \epsilon)) := \begin{cases} \operatorname{argmin}_{(c, f) \in \mathbb{R} \times \mathbb{R}^K} & c^p \\ \text{subject to} & \begin{cases} (c(f))^{1/p} \leq c \\ \|x - \phi(f)\| \leq \epsilon + \delta c. \end{cases} \end{cases} \quad (10)$$

Note that $C_\delta(\phi, x, \epsilon)$ is also the optimal value achieved in (8). In view of this, we slightly abuse the definition (10), and follow the convention that if (8) is infeasible, then $C_\delta(\phi, x, \epsilon) := +\infty$ and $F_\delta(\phi, x, \epsilon) := \emptyset$. In the case of the cost function being strictly convex, the LIP (8) admits a unique optimal solution. If not strictly convex like in the case of sparse encoding, (8) could potentially have multiple solutions depending on the variables $x, \phi, \epsilon, \delta$, and when it does, we see that $F_\delta(\phi, x, \epsilon)$ is a convex set.

2. Recall that a continuous function l defined over an unbounded set U is said to be coercive in the context of an optimization problem, if: $\lim_{\|u\| \rightarrow \infty} l(u) = +\infty$ ($-\infty$), in the context of minimization (maximization) of l and the limit is considered from within the set U .

Definition 2 Let $\phi : \mathbb{R}^K \longrightarrow \mathbb{H}_n$ be a linear map, and let $\epsilon, \delta \geq 0$. A vector $x \in \mathbb{H}_n$ is said to be (ϕ, ϵ, δ) -feasible if $C_\delta(\phi, x, \epsilon) < +\infty$.

Definition 3 For the linear map $\phi : \mathbb{R}^K \longrightarrow \mathbb{H}_n$, non-negative real number δ and the cost function $c : \mathbb{R}^K \longrightarrow [0, +\infty[$ satisfying Assumption 1, let us define

$$\begin{aligned} S_\delta(\phi, 1) &:= \{z \in \mathbb{H}_n : \text{there exists } f \in V_c \text{ satisfying } \|z - \phi(f)\| \leq \delta\}, \text{ and} \\ S_\delta(\phi, r) &:= r \cdot S_\delta(\phi, 1) \text{ for all } r \geq 0. \end{aligned} \quad (11)$$

where V_c is the unit sub level set of the cost function c .

By denoting $S' := \{\phi(f) : f \in V_c\}$, it is clear that S' is the image of the compact and convex set V_c under the linear map ϕ , and is therefore, compact and convex.³ Moreover, since the set $S_\delta(\phi, 1)$ is the image of another linear map: $S' \times B[0, \delta] \ni (z', y) \longmapsto z' + y$, we conclude convexity and compactness of the set $S_\delta(\phi, r)$ for every $r \geq 0$.

It is easy to see that $S_\delta(\phi, 1) = \bigcup_{h \in V_c} B[\phi(h), \delta]$. If $\delta > 0$, the set $S_\delta(\phi, 1)$ has non-empty interior that contains the origin, therefore, $S_\delta(\phi, 1)$ is an absorbing set of \mathbb{H}_n .⁴ If $\delta = 0$, we immediately see on the one hand that $S_0(\phi, r) \subset \text{image}(\phi)$ for every $r \geq 0$. On the other hand, for every $z \in \text{image}(\phi)$ we know that there exists $f \in \mathbb{R}^K$ such that $z = \phi(f)$, therefore, we have $z \in S_0(\phi, (c(f))^{1/p})$. Consequently, we obtain:

$$\lim_{r \rightarrow +\infty} S_\delta(\phi, r) := \bigcup_{r \geq 0} S_\delta(\phi, r) = \begin{cases} \mathbb{H}_n & \delta > 0, \\ \text{image}(\phi) & \delta = 0. \end{cases} \quad (12)$$

Definition 4 The gauge function $\|\cdot\|_\phi : \mathbb{H}_n \longrightarrow [0, +\infty[$ corresponding to the set $S_\delta(\phi, 1)$ is given by

$$\|z\|_\phi := \min \{r \geq 0 : z \in S_\delta(\phi, r)\}. \quad (13)$$

A direct consequence for the gauge function $\|\cdot\|_\phi$ due to (12) is that if $\delta > 0$ then $\|z\|_\phi < +\infty$ for every $z \in \mathbb{H}_n$, and if $\delta = 0$ then $\|z\|_\phi < +\infty$ if and only if $z \in \text{image}(\phi)$. Moreover, due to the set $S_\delta(\phi, r)$ being compact for every $r \geq 0$, the minimization in (13) is always achieved, and we get $z \in S_\delta(\phi, \|z\|_\phi)$ for every $z \in \mathbb{H}_n$ such that $\|z\|_\phi < +\infty$.

The underlying convexity of the linear inverse problem (8) gives rise to an interplay of the convex bodies $B[x, \epsilon]$ and $S_\delta(\phi, r)$. In view of (12), as the scaling factor r is increased, the set $S_\delta(\phi, r)$ grows gradually and intersects with $B[x, \epsilon]$ for every (ϕ, ϵ, δ) -feasible point $x \in \mathbb{H}_n$. We shall see (Lemma 23) that the optimal cost $C_\delta(\phi, x, \epsilon)$ for a given $x \in \mathbb{H}_n$ is proportional to the minimum amount by which the set $S_\delta(\phi, 1)$ needs to be scaled so that it intersects with $B[x, \epsilon]$. This gives rise to the following relation between the optimal cost $C_\delta(\phi, x, \epsilon)$, the gauge function $\|\cdot\|_\phi$, and the set $B[x, \epsilon]$.

3. Considering, for instance, $c(\cdot) = \|\cdot\|_1$ and the linear map ϕ given by the matrix $D = (d_1 \ d_2 \ \dots \ d_K) \in \mathbb{R}^{n \times K}$, we see that V_c is the ℓ_1 -closed ball in \mathbb{R}^K and $S' = \text{conv}(\pm d_i)_{i=1}^K$.

4. A set S is an absorbing set of a vector space H if for every $z \in H$ there exists $r_z \geq 0$ such that $z \in r_z \cdot S$.

Lemma 5 Consider the LIP (8) for the linear map ϕ , cost function c , non-negative real numbers ϵ, δ and $x \in \mathbb{H}_n$. The optimal cost $C_\delta(\phi, x, \epsilon)$ of the LIP (8) and the guage function $\|\cdot\|_\phi$ satisfy

$$(C_\delta(\phi, x, \epsilon))^{1/p} = \min_{y \in B[x, \epsilon]} \|y\|_\phi. \quad (14)$$

2.1 Duality

The guage function $\|\cdot\|_\phi$ gives rise to its corresponding dual function $\|\cdot\|'_\phi : \mathbb{H}_n \rightarrow [0, +\infty[$ defined by:

$$\|\lambda\|'_\phi := \sup_{\|z\|_\phi \leq 1} \langle \lambda, z \rangle = \sup_{z \in S_\delta(\phi, 1)} \langle \lambda, z \rangle. \quad (15)$$

Let $\lambda, y \in \mathbb{H}_n$, we recall that $y \in S_\delta(\phi, \|y\|_\phi)$, and consequently, we have the following Hölder inequality

$$\langle \lambda, y \rangle \leq \sup_{z \in S_\delta(\phi, \|y\|_\phi)} \langle \lambda, z \rangle = \|y\|_\phi \sup_{z \in S_\delta(\phi, 1)} \langle \lambda, z \rangle = \|y\|_\phi \|\lambda\|'_\phi.$$

This gives rise to strong duality between the guage function $\|\cdot\|_\phi$ and its associated dual function $\|\cdot\|'_\phi$ in the following way

$$\|y\|_\phi = \begin{cases} \sup_{\lambda} & \langle \lambda, y \rangle \\ \text{subject to} & \|\lambda\|'_\phi \leq 1. \end{cases} \quad (16)$$

By replacing the guage function $\|\cdot\|_\phi$ in (14) with its equivalent sup formulation provided in (16), we obtain the convex dual of the LIP (8). First, we will define the set $\Lambda_\delta(\phi, x, \epsilon)$ which is the collection of optimal dual variables.

Definition 6 Let the linear map ϕ , a cost function c , and $\epsilon, \delta \geq 0$ be given. Then for every $x \in \mathbb{H}_n \setminus B[0, \epsilon]$ that is (ϕ, ϵ, δ) -feasible, let $\Lambda_\delta(\phi, x, \epsilon) \subset \mathbb{H}_n$ denote the collection of points $\lambda \in \mathbb{H}_n$ that satisfy the following two conditions simultaneously:

- $\|\lambda\|'_\phi = 1$, and
- $\langle \lambda, x \rangle - \epsilon \|\lambda\| = (C_\delta(\phi, x, \epsilon))^{1/p}$.

Theorem 7 Let the linear map ϕ , cost function $c(\cdot)$, real numbers $\epsilon, \delta \geq 0$, and $x \in \mathbb{H}_n$ be given. Consider the linear inverse problem (8) and its convex dual problem:

$$\begin{cases} \sup_{\lambda} & \langle \lambda, x \rangle - \epsilon \|\lambda\| \\ \text{subject to} & \|\lambda\|'_\phi \leq 1. \end{cases} \quad (17)$$

- (i) *Strong Duality:* The supremum value in (17) is finite if and only if x is (ϕ, ϵ, δ) -feasible, and is equal to the optimal cost $(C_\delta(\phi, x, \epsilon))^{1/p}$.
- (ii) *Existence and description of an optimal solution to (17).*

- (a) Irrespective of the value of δ , for any $\epsilon \geq 0$ if $\|x\| \leq \epsilon$, then $\lambda^* = 0$ is an optimal solution.
- (b) If $\|x\| > \epsilon$, the optimization problem (17) admits an optimal solution if and only if the set $\Lambda_\delta(\phi, x, \epsilon)$ defined in Definition 6 is non-empty. If so, then the supremum in (17) is indeed a maximum and it is achieved at λ^* if and only if $\lambda^* \in \Lambda_\delta(\phi, x, \epsilon)$.
- (c) If $\|x\| > \epsilon$ and the set $\Lambda_\delta(\phi, x, \epsilon)$ is empty, the optimization problem (17) does not admit an optimal solution even though the value of the supremum is finite.

Remark 8 We provide a complete description of the set $\Lambda_\delta(\phi, x, \epsilon)$ in Proposition 31. It turns out that the optimal solution to the dual problem (17) can be entirely characterized in terms of the quantities $(C_\delta(\phi, x, \epsilon), F_\delta(\phi, x, \epsilon))$ of the LIP (8) itself. Therefore, if we only have access to a black box that produces an optimal solution to the LIP (8), a corresponding dual optimal solution can be easily computed from the solutions to the LIP itself, and we need not solve the dual problem again separately. This is very helpful in dictionary learning, where the optimal value of dual variables are used to compute a better dictionary.

Remark 9 We look ahead at Proposition 31 and see that the dual problem does not admit any optimal solution only when $\delta = 0$ and $B(x, \epsilon) \cap \text{image}(\phi) = \emptyset$. Interestingly, in that case, we also observe that the corresponding primal problem is not strictly feasible.

2.2 Equivalent Min-Max Problems

Whenever $\|x\| \leq \epsilon$, we immediately see that the pair $\mathbb{R}_+ \times \mathbb{R}^K \ni (c_x, f_x) := (0, 0)$ is feasible for (8). Moreover, since $c(f) > 0$ for every $f \neq 0$ due to inf-compactness and positive homogeneity, we conclude that:

$$C_\delta(\phi, x, \epsilon) = 0, \text{ and } F_\delta(\phi, x, \epsilon) = \{0\} \quad \text{if and only if } \|x\| \leq \epsilon. \quad (18)$$

Therefore, the case: $\|x\| \leq \epsilon$ is uninteresting and inconsequential. In fact, in dictionary learning, since every such sample can be effectively represented by the zero vector, irrespective of the dictionary. The average cost of representation then depends only on the samples that satisfy $\|x\| > \epsilon$. Therefore, for the convex-concave min-max formulation of the LIP (8), we consider only the case when $\|x\| > \epsilon$.

2.2.1 CONSTRAINED FORMULATION

Theorem 10 Let the linear map $\phi : \mathbb{R}^K \rightarrow \mathbb{H}_n$, real numbers $\epsilon, \delta \geq 0$, $q \in]0, 1[$, $r > 0$ and $x \in \mathbb{H}_n \setminus B[0, \epsilon]$ be given. Consider the following min-sup problem:

$$\begin{cases} \min_{h \in V_c} \sup_{\lambda} & r \left(\langle \lambda, x \rangle - \epsilon \|\lambda\| \right)^q - \left(\delta \|\lambda\| + \langle \lambda, \phi(h) \rangle \right) \\ \text{subject to} & \langle \lambda, x \rangle - \epsilon \|\lambda\| > 0. \end{cases} \quad (19)$$

The following assertions hold with regards to the linear inverse problem (8) and the min-sup problem (19).

- (i) The optimal value of (19) is equal to: $s(r, q) (C_\delta(\phi, x, \epsilon))^{\frac{q}{p(1-q)}}$, and therefore finite if and only if x is (ϕ, ϵ, δ) -feasible.⁵
- (ii) Existence and description of a solution to (19) if x is (ϕ, ϵ, δ) -feasible.⁶
- (a) The minimization over variables h in (19) is achieved, and

$$h^* \in \operatorname{argmin}_{h \in V_c} \begin{cases} \sup_{\lambda} & r \left(\langle \lambda, x \rangle - \epsilon \|\lambda\| \right)^q - \left(\delta \|\lambda\| + \langle \lambda, \phi(h) \rangle \right) \\ \text{subject to} & \langle \lambda, x \rangle - \epsilon \|\lambda\| > 0, \end{cases}$$

if and only if $h^* \in \frac{1}{(C_\delta(\phi, x, \epsilon))^{1/p}} \cdot F_\delta(\phi, x, \epsilon)$.

- (b) In addition, the inf-sup problem (19) admits a saddle point solution if and only if the set $\Lambda_\delta(\phi, x, \epsilon)$ is non-empty, then a pair $(h^*, \lambda^*) \in V_c \times \mathbb{H}_n$ is a saddle point solution to (19) if and only if

$$\begin{aligned} h^* &\in \frac{1}{(C_\delta(\phi, x, \epsilon))^{1/p}} \cdot F_\delta(\phi, x, \epsilon) \text{ and} \\ \lambda^* &\in (rq)^{\frac{1}{1-q}} (C_\delta(\phi, x, \epsilon))^{\frac{q}{p(1-q)}} \cdot \Lambda_\delta(\phi, x, \epsilon). \end{aligned}$$

Corollary 11 By considering $r = r(p) := (1+p)p^{\frac{-p}{1+p}}$ and $q = q(p) := \frac{p}{1+p}$, we get

$$C_\delta(\phi, x, \epsilon) = \begin{cases} \min_{h \in V_c} \sup_{\lambda} & r(p) \left(\langle \lambda, x \rangle - \epsilon \|\lambda\| \right)^{q(p)} - \left(\delta \|\lambda\| + \langle \lambda, \phi(h) \rangle \right) \\ \text{subject to} & \langle \lambda, x \rangle - \epsilon \|\lambda\| > 0. \end{cases}$$

In particular, if the cost function $c(\cdot)$ is positively homogeneous of order 1 (like any norm), we have

$$C_\delta(\phi, x, \epsilon) = \begin{cases} \min_{h \in V_c} \sup_{\lambda} & 2\sqrt{\langle \lambda, x \rangle - \epsilon \|\lambda\|} - \left(\delta \|\lambda\| + \langle \lambda, \phi(h) \rangle \right) \\ \text{subject to} & \langle \lambda, x \rangle - \epsilon \|\lambda\| > 0. \end{cases}$$

Remark 12 In the case of signal recovery in ill-posed linear inverse problems, it is assumed that the true signal is a linear combination of only a few elements of some atomic set \mathcal{A} . The signal is recovered by solving the linear inverse problem (8), by considering the cost function c such that $V_c = \operatorname{conv}(\mathcal{A})$ and $p = 1$. If there are sufficiently many linear measurements and if the measurement map ϕ satisfies conditions like the *restricted isometry property* or mutual coherence, the LIP (8) admits a unique solution and the set $F_\delta(\phi, x, \epsilon)$ is then a singleton containing the true signal to be recovered

5. $s(r, q) := \left((1-q)(q^q r)^{\frac{1}{1-q}} \right)$

6. Generally the notion of a solution to a min-max problem $\min_{u \in U} \max_{v \in V} l(u, v)$ is considered as a *saddle point* $(u^*, v^*) \in U \times V$ such that it satisfies

$$l(u^*, v) \leq l(u^*, v^*) \leq l(u, v^*) \quad \text{for all } (u, v) \in U \times V.$$

Remark 13 In several scenarios like the problem of non-negative matrix factorization (Lee and Seung, 2001, 1999; Hoyer, 2004), one has to solve (8) with the additional constraint that $f \in Q$, where $Q \subset \mathbb{R}^K$ is a convex cone. From similar analysis provided in this chapter, it can be easily verified that the resulting equivalent min-max formulation analogous to (19) is

$$\begin{cases} \min_{h \in V_c \cap Q} \sup_{\lambda} & r\left(\langle \lambda, x \rangle - \epsilon \|\lambda\|\right)^q - \left(\delta \|\lambda\| + \langle \lambda, \phi(h) \rangle\right) \\ \text{subject to} & \langle \lambda, x \rangle - \epsilon \|\lambda\| > 0. \end{cases}$$

It should be noted that the definition of the set $S_\delta(\phi, 1)$ then changes to

$$S_\delta(\phi, 1) := \{z \in \mathbb{H}_n : \text{there exists } f \in V_c \cap Q \text{ satisfying } \|z - \phi(f)\| \leq \delta\},$$

and the quantities $\|\cdot\|_\phi$, $\|\cdot\|'_\phi$ and $\Lambda_\delta(\phi, x, \epsilon)$ are accordingly defined w.r.t. the appropriate definition of the set $S_\delta(\phi, 1)$.

Remark 14 The min-max form (19) is convex in f and *strongly*-concave in λ . Moreover, the coupling between the variables f and λ is bi-linear. It is shown in (Du and Hu, 2019) that performing simple gradient ascent-descent updates on the iterates provides exponential convergence under certain assumptions (which hold true in problems like image denoising). Such min-max forms have also been the center of investigation in (Nesterov, 2005), (Nemirovski, 2004), (Chambolle and Pock, 2011), (Zhang and Lin, 2015)

Remark 15 The constraint $\langle \lambda, x \rangle - \epsilon \|\lambda\| > 0$ is inactive at every optimal solution λ^* , and is therefore, insignificant for theoretical purpose.⁷ However, in practice, one must ensure that the inequality is satisfied so that the quantity $(\langle \lambda, x \rangle - \epsilon \|\lambda\|)^q$ is well defined when $q \in]0, 1[$. This is easily ensured by initialising the variable λ so that it satisfies the inequality (for e.g., $\lambda_0 = x$), and then properly selecting the step sizes in subsequent iterations. Suppose if the variable λ is updated as $\lambda \leftarrow \lambda + \alpha \lambda'$, then we must ensure that the inequality $0 < \langle \lambda + \alpha \lambda', x \rangle - \epsilon \|\lambda + \alpha \lambda'\|$ is satisfied. We see that

$$\begin{aligned} \langle \lambda + \alpha \lambda', x \rangle - \epsilon \|\lambda + \alpha \lambda'\| &\geq \langle \lambda + \alpha \lambda', x \rangle - \epsilon (\|\lambda\| + \alpha \|\lambda'\|) \\ &\geq (\langle \lambda, x \rangle - \epsilon \|\lambda\|) + \alpha (\langle \lambda', x \rangle - \epsilon \|\lambda'\|). \end{aligned}$$

If $0 \leq \langle \lambda', x \rangle - \epsilon \|\lambda'\|$, then the required inequality trivially holds, whereas if $0 > \langle \lambda', x \rangle - \epsilon \|\lambda'\|$, then by selecting $\alpha < \frac{\langle \lambda, x \rangle - \epsilon \|\lambda\|}{|\langle \lambda', x \rangle - \epsilon \|\lambda'\|}$, it is easily verified that $0 < \langle \lambda + \alpha \lambda', x \rangle - \epsilon \|\lambda + \alpha \lambda'\|$.

Remark 16 Interchanging the order of optimization in the min-max problem of Corollary 11 using Sion's min-max theorem, and minimizing over h using Lemma 28, we get

$$C_\delta(\phi, x, \epsilon) = \sup_{\lambda} \left\{ r(p) \left(\langle \lambda, x \rangle - \epsilon \|\lambda\| \right)^{\frac{p}{p+1}} - \|\lambda\|'_\phi \right\}.$$

7. If an inequality constraint is satisfied with strict inequality at a point, then it is said to be inactive at that point.

A crucial observation to be made here is that since $\frac{p}{p+1} \in]0, 1[$, the sublinear component $(\langle \lambda, x \rangle - \epsilon \|\lambda\|)^{\frac{p}{p+1}}$ initially grows faster than the linear component $\|\lambda\|'_\phi$ but is eventually overpowered. Alternatively, when $p > 1$, analysis similar to the proof of Lemma 36 shows that

$$C_\delta(\phi, x, \epsilon) = \sup_\lambda \left\{ p(p-1)^{p-1} \left(\langle \lambda, x \rangle - \epsilon \|\lambda\| \right) - \|\lambda\|'_\phi{}^{\frac{p}{p-1}} \right\}. \quad (20)$$

If $\delta = 0$ and we were to replace the error constraint $\phi(f) \in B[x, \epsilon]$ with $\phi(f) \in B_x$, where $B_x \subset \mathbb{H}_n$ is some compact convex subset. By replacing $\langle \lambda, x \rangle - \epsilon \|\lambda\|$ with the quantity $\min_{y \in B_x} \langle \lambda, y \rangle$ in (20), we arrive at the corresponding min-max problem

$$\min_{y \in B_x} \sup_{\lambda \in \mathbb{H}_n} \left\{ p(p-1)^{p-1} \langle \lambda, y \rangle - \|\lambda\|'_\phi{}^{\frac{p}{p-1}} \right\}. \quad (21)$$

This allows us to write equivalent min-max formulations for problems like LASSO as seen in Remark 17.

Remark 17 Another optimization problem of interest and related to the linear inverse problem (8) is:

$$\underset{f \in S}{\text{minimize}} \quad c(x - \psi(f)), \quad (22)$$

where $c(\cdot)$ is a cost function satisfying Assumption 1 with order of homogeneity $p > 1$, $\psi : \mathbb{R}^K \rightarrow \mathbb{H}_n$ is some linear map and S is some compact convex subset of \mathbb{R}^K . A frequent example of such a problem is LASSO, where $c(\cdot) = \|\cdot\|^2$ and $S = \{z : \|z\|_1 \leq \tau\}$.

Defining $y := x - \psi(f)$, the problem (22) simply becomes $\underset{y \in \{x\} - \psi(S)}{\text{minimize}} \quad c(y)$, whose equivalent min-max form using (21) is

$$\min_{y \in \{x\} - \psi(S)} \max_{\lambda \in \mathbb{H}_n} p(p-1)^{p-1} \langle \lambda, y \rangle - \|\lambda\|'_c{}^{\frac{p}{p-1}},$$

where $\|\lambda\|'_c := \max_{h \in V_c} \langle \lambda, h \rangle$. Replacing the minimization over variable y with f , we get the final equivalent min-max form to (22).

$$\min_{f \in S} \max_{\lambda \in \mathbb{H}_n} p(p-1)^{p-1} \left(\langle \lambda, x - \psi(f) \rangle \right) - \|\lambda\|'_c{}^{\frac{p}{p-1}}. \quad (23)$$

Remark 18 When the error constraint $\|x - \phi(f)\| \leq \epsilon + \delta c$ is measured using some norm $\|\cdot\|$, the corresponding min-max problem (analogous to (19)) is written using the corresponding dual norm $\|\cdot\|'$ as

$$\left\{ \min_{h \in V_c} \sup_{\lambda \in \mathbb{H}_n} r(p) \left(\langle \lambda, x \rangle - \epsilon \|\lambda\|' \right)^{q(p)} - \left(\delta \|\lambda\|' + \langle \lambda, \phi(h) \rangle \right) \right\}.$$

2.2.2 UNCONSTRAINED FORMULATION

The reformulation of the LIP (8) as the convex-concave min-max problem (19) provides a way to obtain a solution to the LIP (8) by solving the min-max problem instead. Even though (19) is a convex problem, it is to be observed that the minimization variable h is constrained. Therefore, updating the iterates of the minimization variable involves computing projections at each iterations, which could be an expensive task. So, in cases, where these projections are expensive, we seek to obtain a similar min-max reformulation which bypasses this computational bottleneck due to projections.

Proposition 19 *Let $\epsilon, \delta \geq 0$ and $x \in \mathbb{H}_n \setminus B[0, \epsilon]$, and a positively homogeneous cost function $c(\cdot)$ of order 1 be given. Then the following assertions hold in view of the linear inverse problem (8) and the min-max problem:*

$$\left\{ \begin{array}{l} \min_{f \in \mathbb{R}^K} \sup_{\lambda \in \mathbb{H}_n} c(f)(1 - \delta \|\lambda\|) + (\langle \lambda, x \rangle - \epsilon \|\lambda\|) - \langle \lambda, \phi(f) \rangle \\ \text{subject to} \quad \left\{ \begin{array}{l} \|\lambda\| \leq \frac{1}{\delta} \\ \langle \lambda, x \rangle - \epsilon \|\lambda\| > 0, \end{array} \right. \end{array} \right. \quad (24)$$

- (i) *the optimal value of the min-max problem (24) is equal to $C_\delta(\phi, x, \epsilon)$*
- (ii) *if x is (ϕ, ϵ, δ) -feasible, the minimization over variable f is achieved and $F_\delta(\phi, x, \epsilon)$ is the set of minimizers*
- (iii) *if x is (ϕ, ϵ, δ) -feasible, the min-max problem (24) admits a saddle point solution (f^*, λ^*) if and only if the set $\Lambda_\delta(\phi, x, \epsilon)$ is non-empty, then a pair $(f^*, \lambda^*) \in \mathbb{R}^K \times \mathbb{H}_n$ is saddle point solution if and only if $f^* \in F_\delta(\phi, x, \epsilon)$ and $\lambda^* \in \Lambda_\delta(\phi, x, \epsilon)$.*

Even though the maximization variable is constrained, the reference to the min-max problem (24) as “unconstrained” is due to the fact that projecting onto the feasible set $\{\lambda : \|\lambda\| \leq 1/\delta\}$ is very easy since it only requires normalizing the iterates λ by the factor $\frac{1}{\delta\|\lambda\|}$. Moreover, if $\delta = 0$, the min-max is truly unconstrained, justifying its name.

On the one hand, solving (19) involves projecting the iterates h onto the level set V_c at each iteration, which is generally a demanding task. Whereas, solving (24) requires the relatively easier task of computing gradient of the cost function $c(\cdot)$. However, it is observed that it takes fewer iterations to compute a saddle point solution to the constrained formulation (19) compared to that of the unconstrained formulation (24). Therefore, if we were to find a solution to the LIP (8) by solving the min-max problems (19) or (24), the user has to decide between solving (19) with fewer but expensive iterations or solving (24) with relatively easier but more iterations. This tradeoff depends on the given cost function and could be equally expensive like in the case of minimizing Nuclear norm.

Remark 20 Observe that the order of homogeneity p in the LIP (8) is assumed to be 1 in Proposition 19. If $p > 1$ and when $\delta = 0$, the equivalent unconstrained min-max reformulation similar to (24) is

$$\left\{ \begin{array}{l} \min_{f \in \mathbb{R}^K} \sup_{\lambda \in \mathbb{H}_n} c(f) + \left(p^{\frac{1+2p}{p}} \right) (\langle \lambda, x \rangle - \epsilon \|\lambda\|) - \langle \lambda, \phi(f) \rangle. \end{array} \right. \quad (25)$$

Remark 21 If $\delta = 0$ and we were to replace the error constraint $\phi(f) \in B[x, \epsilon]$ with $\phi(f) \in B_x$, where $B_x \subset \mathbb{H}_n$ is some compact convex subset. The min-max problem written using (24) is

$$\begin{cases} \min_{\substack{f \in \mathbb{R}^K \\ y \in B_x}} \sup_{\lambda \in \mathbb{H}_n} & c(f) - \langle \lambda, \phi(f) \rangle + \left(p^{\frac{1+2p}{p}} \right) \langle \lambda, y \rangle . \end{cases}$$

2.3 Algorithms

We propose to solve the LIP (8) by computing saddle point solutions using existing algorithms to solve the min-max problems (19) or (24). There are many techniques available for solving such min-max problems, notably among them are the vanilla Gradient Descent Ascent (GDA), Optimistic Gradient Descent Ascent (OGDA), Proximal Point (PP) and Extra Gradient (EG) algorithms. A quick review of these algorithms can be found in (Mokhtari et al., 2019). The performance of the algorithm to solve LIP depends on the convergence attributes of the algorithm chosen to solve the min-max problems.

For the constrained formulation (19), Algorithm 1 performs projected gradient descent on the minimization variable h for each iteration, and it computes gradients by solving the maximization over λ by keeping the variable h fixed. This can also be done alternatively by performing gradient ascent on the maximization variable over a faster time scale and performing projected gradient descent on the minimization variable over a slower time scale. The unconstrained min-max problem is solved in Algorithm 2 by performing momentum based Optimistic Gradient Descent Ascent (OGDA) update.⁸ A comparison between Algorithms 1 and 2 for ℓ_1 -minimization problems may be found in Section 2.4.1. We would like to emphasise that the particular algorithms 1 and 2 to solve LIP provided in this chapter are for representation purpose only. In practice, depending on specifics of the given LIP the user has to select the appropriate saddle point seeking algorithm to solve the equivalent min-max problems.

2.4 Applications to Standard Problems

In this section, we will discuss two linear inverse type problems namely *Basis Pursuit Denoising* (BPDN) and the *Quadratic Program* (QP). We will discuss the corresponding min-max forms for these problems, and implement the corresponding algorithms on image denoising problems. Finally, we will also discuss how the min-max forms help us in overcoming the ill-posedness of the dictionary learning problem.

8. In the algorithms, ϕ^a indicates the conjugate of the linear map ϕ , $\pi_c(\cdot)$ is the projection operator onto the level set V_c , and $\partial c(f)$ is the subdifferential of the cost function c evaluated at f (if the function c is differentiable at f , we slightly abuse the notation and use the same notation $\partial c(f)$ to refer to the gradient).

Algorithm 1: Projected gradient descent algorithm for constrained min-max problem (19).

Input: Problem data: $x, \phi, \epsilon, \delta, c$.

Output: An optimal solution $f \in F_\delta(\phi, x, \epsilon)$ and the optimal value $C_\delta(\phi, x, \epsilon)$.

1 Proceed only if $\|x\| > \epsilon$, else output 0.

2 Initialize $t = 0, h_0$ and λ_0 .

3 **Iterate till convergence**

 Initialise $\lambda = \lambda_t$ and iterate M times

$$\lambda \leftarrow \lambda + \alpha \left(\frac{p^{\frac{1}{1+p}} (x - \frac{\epsilon}{\|\lambda\|} \lambda)}{(\langle \lambda, x \rangle - \epsilon \|\lambda\|)^{\frac{1}{1+p}}} - \frac{\delta}{\|\lambda\|} \lambda - \phi(h_t) \right)$$

 Update : $h_{t+1} = \pi_c(h_t + \beta_t(\phi^a(\lambda)))$ and $\lambda_{t+1} = \lambda$

$t \leftarrow t + 1$

4 **Repeat**

5 **Output:** $C_\delta(\phi, x, \epsilon) = \langle \lambda_t, \phi(h_t) \rangle$ and $f = C_\delta(\phi, x, \epsilon) \cdot h_t$.

Algorithm 2: OGDA algorithm for unconstrained min-max problem (24) when $\delta = 0$.

Input: Problem data: x, ϕ, ϵ, c .

Output: An optimal solution $f \in F_\delta(\phi, x, \epsilon)$.

1 Proceed only if $\|x\| > \epsilon$, else output 0.

2 Initialize $t = 0, (f_0, \lambda_0)$ and $(\Gamma_{-1}, \gamma_{-1})$.

3 **Iterate till convergence**

 Compute $\gamma_t \in (\partial c(f_t) - \phi^a(\lambda_t))$ and $\Gamma_t = (x - \phi(f_t) - \frac{\epsilon}{\|\lambda_t\|} \lambda_t)$.

 Update :

$$f_{t+1} = f_t - \alpha'_t (2\gamma_t - \gamma_{t-1})$$

$$\lambda_{t+1} = \lambda_t + \alpha'_t (2\Gamma_t - \Gamma_{t-1})$$

$t \leftarrow t + 1$

4 **Repeat**

5 **Output:** f_t .

2.4.1 BASIS PURSUIT DENOISING (BPDN)

One of the most practical example of an LIP is the classical *Basis Pursuit Denoising* problem (Elad and Aharon, 2006; Candès and Wakin, 2008), which arises in various scenarios of compressed sensing and image processing like denoising, deblurring etc.

$$\begin{cases} \text{minimize} & \|f\|_1 \\ f \in \mathbb{R}^K & \\ \text{subject to} & \|x - \phi(f)\|_2 \leq \epsilon. \end{cases} \quad (26)$$

In such image processing problems, instead of solving the problem directly on the entire image, it is often done on a collection of smaller patches (typically of size 8×8) that cover the entire image. Since natural images are reasonably sparse in 2d-DCT (Discrete Cosine Transform) basis, a common choice for the linear map ϕ is the inverse 2d-DCT operator for 8×8 images.

We implement BPDN based image denoising on two images by solving (26) for all non overlapping patches of size 8×8 . Figure 1 shows the denoising results for the standard cameraman image, which is of size 256×256 . From left to right, we have the original image, noisy image and the recovered image in order. The noisy image is obtained by adding a mean zero Gaussian noise of standard deviation 0.0065 using `imnoise` function in MATLAB resulting in a PSNR (Peak Signal to Noise Ratio) of 22.0741dB. To recover the image, BPDN was solved with $\epsilon = 0.3$ for every non-overlapping 8×8 patch using Algorithm 1 to compute the saddle point of the equivalent constrained min-max problem

$$\begin{cases} \min_{\|h\|_1 \leq 1} \sup_{\lambda} & 2\sqrt{\lambda^\top x - \epsilon \|\lambda\|_2} - \lambda^\top \phi(h) \\ \text{subject to} & \lambda^\top x - \epsilon \|\lambda\|_2 > 0. \end{cases} \quad (27)$$

The recovered image has a PSNR value of 26.8119dB.



Figure 1: In order from left to right, we have the original image, noisy image, and the recovered image. The image recovery is done by employing Algorithm 1 to solve the BPDN problem (26) on each 8×8 patch. The PSNR value of the noisy image is 22.0741dB, and that of the recovered image is 26.8119dB, both w.r.t. the original image.

Similarly, in Figure 2 the results for denoising the flower image are shown, where the noisy image is obtained by adding a mean zero Gaussian noise of standard deviation 0.005, resulting in a PSNR of 23.0954dB. To denoise the image, BPDN was solved with $\epsilon = 0.385$ for every non-overlapping 8×8 patch using Algorithm 2 to compute saddle points of the

equivalent unconstrained min-max problem

$$\min_f \sup_\lambda \left\{ \|f\|_1 + \lambda^\top (x - \phi(f)) - \epsilon \|\lambda\|_2 \right\}. \quad (28)$$

The recovered image has a PSNR value of 28.5362dB.

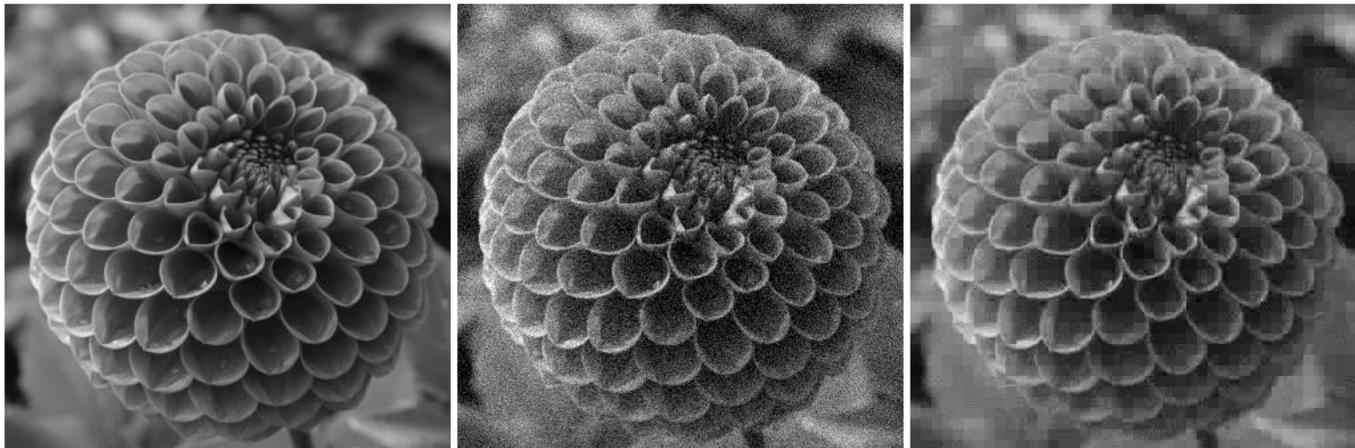


Figure 2: In order from left to right, we have the original image, noisy image, and the recovered image. The image recovery is done by employing Algorithm 2 to solve the BPDN problem (26) on each 8×8 patch. The PSNR value of the noisy image is 23.0954dB, and that of the recovered image is 28.5362dB, both w.r.t. the original image.

For comparison purpose, we solve a BPDN problem for a randomly selected 8×8 patch from the noisy flower image in Figure 2 via both Algorithm 1 and 2. In Algorithm 1, we choose the step size sequence $\alpha_t = 5/(20 + t)$, $\epsilon = 0.385$ and for initialisation, $h_0 = \frac{\phi^+(x)}{\|\phi^+(x)\|_1}$ and $f_0 = x$, where ϕ^+ is the pseudo inverse of ϕ . To show convergence to a saddle point in Figure 3, we plot the quantities

$$\begin{aligned} G_t &= \left\| \frac{(x - \frac{\epsilon}{\|\lambda_t\|} \lambda_t)}{\sqrt{\langle \lambda_t, x \rangle - \epsilon \|\lambda_t\|}} - \phi(h_{t-1}) \right\| \\ g_t &= \|\phi^a(\lambda_t)\|_\infty - \langle \lambda_t, \phi(h_t) \rangle. \end{aligned} \quad (29)$$

Note that $G_t = 0$ guarantees the maximization condition, and $g_t = 0$ guarantees the minimization condition in the constrained min-max problem (27). Therefore, if $g_t = G_t = 0$ for some t , it implies that (λ_t, h_t) is a saddle point to (27). Therefore, the sequences $(g_t)_t$ and $(G_t)_t$ act as certificates for the saddle point condition for the min-max problem (27).

Similarly, to compute a saddle point of the unconstrained min-max problem (28) via Algorithm 2, we use the step size sequence $\alpha_t = 2.5/(10 + t)$, $\epsilon = 0.385$ and for initialisation,

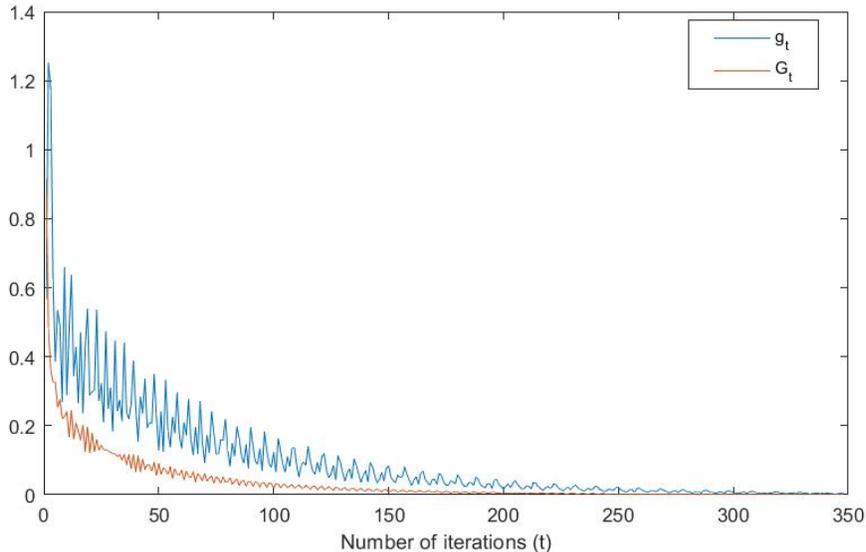


Figure 3: Saddle point certificates to solve unconstrained min-max problem (27) via Algorithm 1.

$f_0 = \phi^+(x)$ and $\lambda_0 = x$. To show convergence to a saddle point in Figure 4, we have plot the certificates

$$\begin{aligned} G'_t &= \alpha_t \|2\Gamma_t - \Gamma_{t-1}\| \\ g'_t &= \alpha_t \|2\gamma_t - \gamma_{t-1}\|. \end{aligned} \quad (30)$$

Evidently, the constrained formulation requires fewer iterations to converge to a reasonable solution, in comparison to the number of iterations required to solve (28). However, each iteration to solve (27) involves computing projections onto ℓ_1 -ball, which makes each iteration expensive.

2.4.2 PROJECTION ONTO CONVEX SETS AND QUADRATIC PROGRAMS

Another optimization problem that arises regularly is the Quadratic Program (QP). Particularly, the problem of projecting a given point x onto some given compact-convex set S . Projection of points onto convex sets arises in almost every practical optimization problem where the iterates have to be projected onto the respective feasible sets after they are updated using a gradient descent like technique to minimize the cost. We consider the following QP

$$\underset{y \in S}{\text{minimize}} \quad \langle y - x, Q(y - x) \rangle, \quad (31)$$

where Q is some given positive definite matrix. In terms of new variable $f := y - x$, the QP (31) becomes

$$\underset{f \in S - \{x\}}{\text{minimize}} \quad \langle f, Qf \rangle, \quad (32)$$

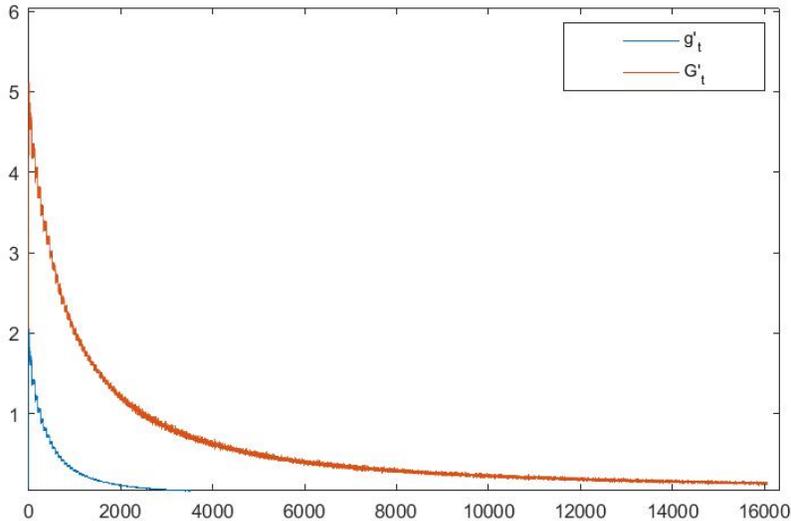


Figure 4: Saddle point certificates to solve unconstrained min-max problem (28) via Algorithm 2.

which is in the form of the LIP (8) with $\delta = 0$, $c(f) = \langle f, Qf \rangle$, ϕ being the identity map and the constraint $f \in S - \{x\}$ instead of $f \in B[x, \epsilon]$. Using the min-max reformulation (19) of the LIP, and replacing the quantity $\langle \lambda, x \rangle - \epsilon \|\lambda\|$ with $\min_{y' \in S - \{x\}} \langle \lambda, y' \rangle$ we obtain the following equivalent min-max formulation for the QP (31).

$$\begin{cases} \min_{\langle h, Qh \rangle \leq 1} \max_{\lambda} & 1.889 \left(\min_{y \in S} \langle \lambda, y - x \rangle \right)^{2/3} - \langle \lambda, h \rangle \\ \text{subject to} & 0 < \min_{y \in S} \langle \lambda, y - x \rangle. \end{cases} \quad (33)$$

Since the set S is compact, the QP (31) always admits a solution and the corresponding min-max problem (33) admits a saddle point solution (h^*, λ^*) . From first order conditions, we know that the optimal value of the min-max problem is $0.5 \langle \lambda^*, h^* \rangle$. Then, simple algebra shows that the optimal solution f^* to (32) is given by $(0.5 \langle \lambda^*, h^* \rangle) h^*$, and the optimal solution y^* to the QP (31) is $y^* = x + f^*$.

The advantage lies in the fact that the projection map $\pi_Q : \mathbb{H}_n \rightarrow \mathbb{H}_n$ onto the level set $\{h : \langle h, Qh \rangle \leq 1\}$ is a lot simpler than projecting onto the given set S itself. For example, if Q is identity matrix, then $\pi_Q(h)$ simply normalizes h to have unit norm. Since in most relevant cases, the function $\lambda \mapsto \min_{y \in S} \langle \lambda, y \rangle$ is not differentiable, the step sizes α have to be diminishing for the subgradient descent-ascent to converge.

Alternatively, since the minimization $\min_{\langle h, Qh \rangle \leq 1} \langle \lambda, h \rangle$ admits a unique solution for every λ , the saddle point solution to the min-max problem can be computed by solving the

Algorithm 3: Projected gradient descent algorithm for constrained min-max problem (19).

Input: Problem data: x , Q and the set S .

Output: The solution y^* to QP (31).

1 Initialize $t = 0$ and λ_0 .

2 **Iterate till convergence**

Compute $y_t \in \operatorname{argmin}_{y \in S} \langle \lambda_t, y \rangle$, and

$$\lambda_{t+1} = \lambda_t + \alpha_t \left(\frac{1.26(y_t - x)}{\langle \lambda_t, y_t - x \rangle^{1/3}} - h_t \right)$$

$$h_{t+1} = \pi_Q(h_t + \alpha_t \lambda_t)$$

$t \leftarrow t + 1$

3 **Repeat**

4 **Output:** $y^* = x + \frac{1}{2}Q^{-1}\lambda_t$.

following maximization problem alone.

$$\sup_{\lambda} 1.889 \left(\min_{y \in S} \langle \lambda, y - x \rangle \right)^{2/3} - \left\| Q^{-\frac{1}{2}} \lambda \right\|, \quad (34)$$

using subgradient ascent type algorithms. If (h^*, λ^*) is a saddle point solution to the min-max problem (33), then

$$h^* = \frac{Q^{-1}\lambda^*}{\sqrt{\langle \lambda^*, Q^{-1}\lambda^* \rangle}} = \operatorname{argmax}_{\langle h, Qh \rangle \leq 1} \langle \lambda^*, h \rangle.$$

Finally, the optimal solution y^* to the QP (31) is $y^* = x + \frac{1}{2}Q^{-1}\lambda^*$.

As an exercise, we solve the orthogonal projection problem of projecting a point x onto the ℓ_1 -ball, which if we recall, arises often while using Algorithm 1 to solve the LIP when $c(\cdot) = \|\cdot\|_1$. The point x is drawn uniformly randomly from the ℓ_∞ -ball of \mathbb{R}^{1000} , and we solve (31) with Q being the identity operator using Algorithm 3. Figure 5 shows the progressive distance between the original point x and the computed projections $x + 0.5\lambda_t$ at each iteration of Algorithm 3.

2.4.3 THE DICTIONARY LEARNING PROBLEM

The setup is that every vector $x \in \mathbb{H}_n$ is *encoded* as a vector $f(x)$ in \mathbb{R}^K via the *encoder* map $f : \mathbb{H}_n \rightarrow \mathbb{R}^K$. We shall refer to $f(x)$ as the *representation* of x under the encoder f . The *reconstruction* of the encoded samples from the representation $f(x)$ is done by taking the linear combination $\sum_{i=1}^K f_i(x)d_i$ with some standard collection of vectors $D := (d_1 \ d_2 \ \cdots \ d_K)$ referred to as the *dictionary*. Since the reconstruction has to be a good representative of the true vector x , we constraint the error $\|x - Df(x)\|$ to be small.

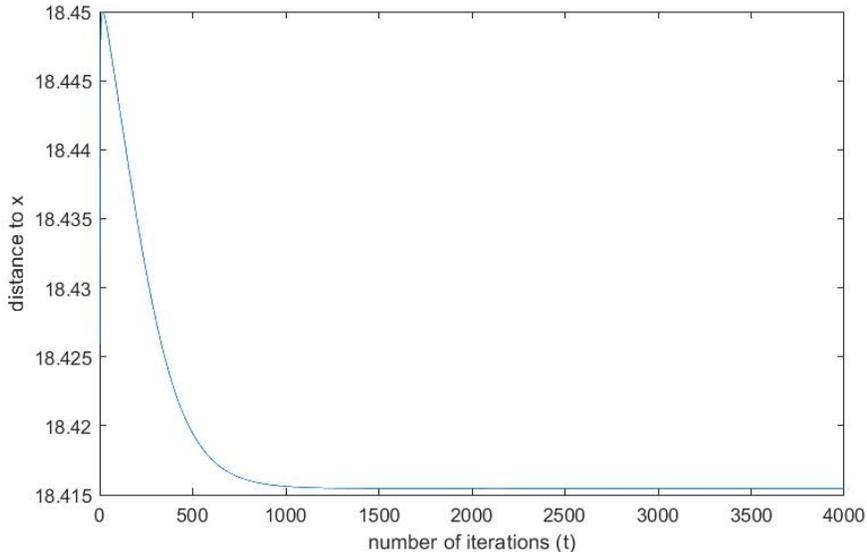


Figure 5: For projecting a point x onto the ℓ_1 -ball, the plot shows the distance between the iterates $(x + 0.5\lambda_t)$ in Algorithm 3 to the original point x .

Given a dictionary D , we encode every vector x by solving the LIP (8) with an appropriate cost function c . This cost function determines the desirable characteristics in the representation. In other words, the optimal encoder map $f_D : \mathbb{H}_n \rightarrow \mathbb{R}^K$ corresponding to the dictionary D is such that $f_D(x) \in F_\delta(D, x, \epsilon)$ for every x . Our objective is to find dictionaries such that the corresponding encoder map f_D has desirable features like sparsity, robustness with respect to loss of coefficients etc., in the representation. We refer to the task of finding such a dictionary as the *dictionary learning problem*.

Formally, let \mathbf{P} be a distribution on \mathbb{H}_n and X be a \mathbf{P} distributed random variable. Let $c : \mathbb{R}^K \rightarrow [0, +\infty[$ be a given cost function that satisfies Assumption 1, $\epsilon : \mathbb{H}_n \rightarrow [0, +\infty[$ be a given error threshold function and δ be a non-negative real number. Given a dictionary D , since the random variable X is encoded as $f_D(X) \in F_\delta(D, X, \epsilon(X))$, we consider the cost incurred to encode to be $C_\delta(D, X, \epsilon(X))$. Our objective is to find a dictionary that facilitates optimal encoding of the data, which are the samples drawn from \mathbf{P} . Therefore, we consider the following dictionary learning problem :

$$\underset{D \in \mathcal{D}}{\text{minimize}} \quad \mathbb{E}_{\mathbf{P}} [C_\delta(D, X, \epsilon(X))], \quad (35)$$

where $\mathcal{D} \subset \mathbb{R}^{n \times K}$ is some known compact convex subset.

For a large integer T , let $(x_t)_{t=1}^T$ be a collection of samples drawn from the distribution \mathbf{P} . Let us consider the dictionary learning problem for the sampled data, given by:

$$\underset{D \in \mathcal{D}}{\text{minimize}} \quad \frac{1}{T} \sum_{t=1}^T C_\delta(D, x_t, \epsilon_t), \quad (36)$$

where $\epsilon_t = \epsilon(x_t)$ for all t . For the special case of $\delta = 0$, the dictionary learning problem (36) can be restated using the definition of the encoding cost $C_\delta(D, x_t, \epsilon(x_t))$ in the more conventional form as:

$$\left\{ \begin{array}{l} \underset{D, (f_t)_t}{\text{minimize}} \quad \frac{1}{T} \sum_{t=1}^T c(f_t) \\ \text{subject to} \quad \left\{ \begin{array}{l} D \in \mathcal{D}, \\ f_t \in \mathbb{R}^K, \\ \|x_t - Df_t\| \leq \epsilon_t \text{ for all } t = 1, 2, \dots, T. \end{array} \right. \end{array} \right. \quad (37)$$

The dependence of the encoding cost $C_\delta(D, x, \epsilon)$ on the dictionary variable D is not immediately evident. Therefore, it is replaced in the dictionary learning problem (36) with the min-max problem provided in Corollary 11 to obtain

$$\min_{D \in \mathcal{D}} \min_{(h_t)_t \subset V_c} \left\{ \begin{array}{l} \sup_{(\lambda_t)_t} \quad \frac{1}{T} \sum_{t=1}^T \left(r(p) (\langle \lambda_t, x_t \rangle - \epsilon_t \|\lambda_t\|)^{\frac{p}{1+p}} - (\delta \|\lambda_t\| + \langle \lambda_t, Dh_t \rangle) \right) \\ \text{s.t.} \quad \langle \lambda_t, x_t \rangle - \epsilon_t \|\lambda_t\| > 0, \end{array} \right.$$

where $r(p) = (1+p)p^{\frac{-p}{1+p}}$. The dictionary learning problem (36) is then solved by alternating the optimization over D and $(h_t)_t$ keeping the other one fixed. It is to be noted that each of these optimization problem is a min-max problem in variables $(D, (\lambda_t)_t)$ and $((h_t)_t, (\lambda_t)_t)$ respectively. In particular, for a given sequence $(h_t)_t \subset V_c$, the dictionary is updated by solving the following min-max problem

$$\left\{ \begin{array}{l} \min_{D \in \mathcal{D}} \sup_{(\lambda_t)_t} \quad \frac{1}{T} \sum_{t=1}^T \left(r(p) (\langle \lambda_t, x_t \rangle - \epsilon_t \|\lambda_t\|)^{\frac{p}{1+p}} - (\delta \|\lambda_t\| + \langle \lambda_t, Dh_t \rangle) \right) \\ \text{subject to} \quad \langle \lambda_t, x_t \rangle - \epsilon_t \|\lambda_t\| > 0. \end{array} \right.$$

It is shown in (Sheriff and Chatterjee, 2020) that if $\delta > 0$ the above min-max problem always admits a saddle point solution. Moreover, we observe the objective function of the min-max problem is linear w.r.t. the dictionary variable D and concave w.r.t. λ . Thus, the saddle point solution can be computed efficiently by simple ascent-descent type iterations. The novelty and the convergence attributes of learning a dictionary to solve (36) can be attributed to the reformulations (19), (24) of the LIP provided in this article.

3. Theory, Discussion, and Proofs.

In this section we investigate the linear inverse problem (8) in detail and its dual with special emphasis on the underlying convex geometry. Based on the principle of separation of convex bodies by linear functionals, we obtain the dual problem (17) of the linear inverse problem which then leads to the convex-concave min-max problem (17). We later provide the proof of Theorem 10 establishing that the optimal value of this min-max problem is proportional to the optimal cost $C_\delta(\phi, x, \epsilon)$ of the LIP (8).

Lemma 22 For a linear map $\phi : \mathbb{R}^K \longrightarrow \mathbb{H}_n$ and $\delta, r \geq 0$, we have

$$S_\delta(\phi, r) = \{z \in \mathbb{H}_n : C_\delta(\phi, z, 0) \leq r^p\}. \quad (38)$$

Proof On the one hand, it follows from the Definition 3 that for every $z \in S_\delta(\phi, r)$, there exists $f_z \in \mathbb{R}^K$ such that $c(f_z) \leq r^p$ and $\|z - \phi(f_z)\| \leq \delta r$. Thus, considering $\epsilon = 0$ in (8), we see that the pair (r, f_z) is a feasible point and hence we have $C_\delta(\phi, z, 0) \leq r^p$.

On the other hand, if $z \in \mathbb{H}_n$ is such that $C_\delta(\phi, z, 0) \leq r^p$, we know that there exists a pair $(c_z, f_z) \in \mathbb{R}_+ \times \mathbb{R}^K$ such that $c_z^p = C_\delta(\phi, z, 0) \leq r^p$ and satisfies the following:

- $c(f_z) \leq c_z^p \leq r^p$, and
- $\|z - \phi(f_z)\| \leq 0 + \delta c_z \leq \delta r$.

It then immediately follows that for every $z \in \mathbb{H}_n$ satisfying $C_\delta(\phi, z, 0) \leq r^p$, we have the membership $z \in S_\delta(\phi, r)$. Collecting the two assertions we arrive at (38). \blacksquare

Lemma 23 For a given linear map $\phi : \mathbb{R}^K \longrightarrow \mathbb{H}_n$ and non-negative real numbers ϵ, δ , let $x \in \mathbb{H}_n$ be (ϕ, ϵ, δ) -feasible in the sense of Definition 2, then we have

$$C_\delta(\phi, x, \epsilon) = \begin{cases} \min_{r \geq 0} & r^p \\ \text{subject to} & S_\delta(\phi, r) \cap B[x, \epsilon] \neq \emptyset. \end{cases} \quad (39)$$

Proof Let $r \geq 0$ be such that $S_\delta(\phi, r) \cap B[x, \epsilon] \neq \emptyset$. Then on the one hand, there exists $y_r \in B[x, \epsilon]$ and $f_r \in \mathbb{R}^K$ such that $\|y_r - \phi(f_r)\| \leq \delta r$ and $c(f_r) \leq r^p$. From this we get

$$\|x - \phi(f_r)\| \leq \|x - y_r\| + \|y_r - \phi(f_r)\| \leq \epsilon + \delta r,$$

which implies that the pair (r, f_r) is feasible for (8), and as a result we get $C_\delta(\phi, x, \epsilon) \leq r^p$. By minimizing over $r \geq 0$ such that $S_\delta(\phi, r) \cap B[x, \epsilon] \neq \emptyset$ we get our first inequality:

$$C_\delta(\phi, x, \epsilon) \leq \begin{cases} \inf_{r \geq 0} & r^p \\ \text{subject to} & S_\delta(\phi, r) \cap B[x, \epsilon] \neq \emptyset. \end{cases}$$

On the other hand, for every pair (r', f') that is feasible for (8), by defining $y := x \mathbf{1}_{\{0\}}(\epsilon) + \frac{\epsilon \phi(f') + \delta r' x}{\epsilon + \delta r'} \mathbf{1}_{]0, +\infty[}(\epsilon)$ we shall establish that $y \in B[x, \epsilon] \cap S_\delta(\phi, r')$.

Whenever $\epsilon = 0$ we have $y = x$, and from the feasibility of the pair (r', f') it easily follows that $\|y - \phi(f')\| = \|x - \phi(f')\| \leq \epsilon + \delta r' = \delta r'$ and $c(f') \leq r'^p$. Thus, $C_\delta(y, \phi, 0) \leq r'^p$ and from (38) the membership $y \in S_\delta(\phi, r')$ holds. Moreover, if $\epsilon = 0$, we also see that $y = x = B[x, 0]$. Therefore, $y \in B[x, \epsilon] \cap S_\delta(\phi, r')$, and the intersection is non-empty.

When $\epsilon > 0$, we see that

$$\begin{aligned} \|x - y\| &= \left\| x - \frac{\epsilon \phi(f') + \delta r' x}{\epsilon + \delta r'} \right\| = \frac{\epsilon}{\epsilon + \delta r'} \|x - \phi(f')\| \leq \epsilon, \text{ and} \\ \|y - \phi(f')\| &= \left\| \frac{\epsilon \phi(f') + \delta r' x}{\epsilon + \delta r'} - \phi(f') \right\| = \frac{\delta r'}{\epsilon + \delta r'} \|x - \phi(f')\| \leq \delta r'. \end{aligned}$$

These inequalities, along with the fact that $c(f') \leq r'^p$ imply that $y \in B[x, \epsilon] \cap S_\delta(\phi, r')$ and in particular that $B[x, \epsilon] \cap S_\delta(\phi, r') \neq \emptyset$. As a consequence, the inequality:

$$r' \geq \begin{cases} \inf_{r \geq 0} & r \\ \text{subject to} & S_\delta(\phi, r) \cap B[x, \epsilon] \neq \emptyset, \end{cases}$$

holds for every pair (r', f') that is feasible for (8). By minimizing over all such feasible pairs (r', f') , we obtain the converse inequality

$$C_\delta(\phi, x, \epsilon) \geq \begin{cases} \inf_{r \geq 0} & r^p \\ \text{subject to} & S_\delta(\phi, r) \cap B[x, \epsilon] \neq \emptyset. \end{cases}$$

Therefore,

$$C_\delta(\phi, x, \epsilon) = \begin{cases} \inf_{r \geq 0} & r^p \\ \text{subject to} & S_\delta(\phi, r) \cap B[x, \epsilon] \neq \emptyset. \end{cases}$$

We complete the proof by concluding that the infimum above is indeed a minimum since $B[x, \epsilon]$ and $S_\delta(\phi, r)$ are compact subsets of \mathbb{H}_n . \blacksquare

Remark 24 An interesting viewpoint to take from this in dictionary learning problem is that every dictionary D gives rise to an atomic set $S_\delta(D, 1)$, and the encoding cost $C_\delta(D, x, \epsilon)$ of a vector x is proportional to the *approximate* Minkowski gauge function with respect to this set.⁹ The corresponding dictionary learning problem can be viewed as the task of finding a ‘good’ atomic set arising from a dictionary.

3.1 Intersection of the Convex Bodies.

Lemma 23 gives us the first required connection between the LIP (8) and the underlying convex geometry. It asserts that the value $(C_\delta(\phi, x, \epsilon))^{1/p}$ is the minimum amount by which the set $S_\delta(\phi, 1)$ has to be scaled linearly so that it intersects with $B[x, \epsilon]$. To this end, let us define:

$$S_\delta(\phi, x, \epsilon) := (C_\delta(\phi, x, \epsilon))^{1/p} \cdot S_\delta(\phi, 1).$$

We observe that both the sets $S_\delta(\phi, x, \epsilon)$ and $B[x, \epsilon]$ are compact and convex, and due to this, we have the following intersection lemma.

Lemma 25 *Let $x \in \mathbb{H}_n$ be (ϕ, ϵ, δ) -feasible in the sense of Definition 2. Let $(\mathbf{c}_x, f_x) \in [0, +\infty[\times \mathbb{R}^K$ be an optimal solution to the LIP (8), i.e., $\mathbf{c}_x = (C_\delta(\phi, x, \epsilon))^{1/p}$ and $f_x \in F_\delta(\phi, x, \epsilon)$. Then the sets $B[x, \epsilon]$ and $S_\delta(\phi, x, \epsilon)$ intersect at a unique point y^* given by:*

$$B[x, \epsilon] \cap S_\delta(\phi, x, \epsilon) =: y^* = x \mathbb{1}_{\{0\}}(\epsilon) + \frac{\epsilon \phi(f_x) + \delta \mathbf{c}_x x}{\epsilon + \delta \mathbf{c}_x} \mathbb{1}_{]0, +\infty[}(\epsilon). \quad (40)$$

In addition, we also assert that

9. We say ‘‘approximate’’ in the sense that we do not scale the atomic set $S_\delta(D, 1)$ so as to absorb x . Instead, we scale it only until it intersects with a given neighborhood of x .

- (i) whenever, $\|x\| > \epsilon$, every $f_x \in F_\delta(\phi, x, \epsilon)$ satisfies $\|x - \phi(f_x)\| = \epsilon + \delta c_x$;
- (ii) for every $f_x, g_x \in F_\delta(\phi, x, \epsilon)$, we have $\phi(f_x) = \phi(g_x)$.

Remark 26 In dictionary learning, for a given dictionary D , since every sample vector $x \in \mathbb{H}_n$ is represented by some vector $f_D(x) \in F_\delta(D, x, \epsilon)$, the representation is not unique whenever the set $F_\delta(D, x, \epsilon)$ is not a singleton. In such situations, even though the representation need not be unique, we emphasize that the reconstruction $x_{\text{rec}} := Df_D(x)$ of the vector x obtained from its representation $f_D(x)$ is unique.

Remark 27 When $x \notin B[0, \epsilon]$ since $c_x > 0$, it is easily verified that the unique point of intersection y^* in Lemma 25 can also be written as:

$$y^* = \phi(f_x) + \frac{\delta c_x}{\epsilon + \delta c_x} (x - \phi(f_x)) \mathbf{1}_{]0, +\infty[}(\delta).$$

Proof [Lemma 25] We note that if $\epsilon = 0$, $B[x, \epsilon] = x$, and since by definition, the set $S_\delta(\phi, x, \epsilon)$ intersects with $B[x, \epsilon]$. The intersection happens at the point x which is unique. We shall establish (40) by considering the remaining cases.

- $0 < \|x\| \leq \epsilon$: From (18), we know that $C_\delta(\phi, x, \epsilon) = 0$ and $F_\delta(\phi, x, \epsilon) = \{0\}$. This implies that $S_\delta(\phi, x, \epsilon) = \{0\}$. In addition, we see that $0 \in B[x, \epsilon]$ whenever $\|x\| \leq \epsilon$. As a result, we obtain that $B[x, \epsilon] \cap S_\delta(\phi, x, \epsilon) = \{0\}$. Now, by using the fact that (c_x, f_x) is an optimal solution to the LIP (8) if and only if $(c_x, f_x) = (0, 0)$, we see that y^* in (40) evaluates to 0 confirming (40).
- $0 < \epsilon < \|x\|$: We shall prove by contradiction that the sets $B[x, \epsilon]$ and $S_\delta(\phi, x, \epsilon)$ intersect at a unique point. Let $y_1 \neq y_2$ be such that $y_1, y_2 \in B[x, \epsilon] \cap S_\delta(\phi, x, \epsilon)$. Since $B[x, \epsilon]$ is a strictly convex set, $\frac{1}{2}(y_1 + y_2) \in B(x, \epsilon)$. However, since $B(x, \epsilon)$ is an open set, one can find $\rho > 0$ such that $B[\frac{1}{2}(y_1 + y_2), \rho] \subset B(x, \epsilon)$. Since $0 \notin B(x, \epsilon)$, we conclude that $2\rho < \|y_1 + y_2\|$; Defining $\theta := \left(1 - \frac{2\rho}{\|y_1 + y_2\|}\right)$, we see that $\theta \in]0, 1[$. It is easily verified that $\|\frac{1}{2}(y_1 + y_2) - \frac{\theta}{2}(y_1 + y_2)\| = \rho$, which leads us to the first inclusion $\frac{\theta}{2}(y_1 + y_2) \in B[\frac{1}{2}(y_1 + y_2), \rho] \subset B[x, \epsilon]$. In addition, we note that the set $S_\delta(\phi, x, \epsilon)$ is also convex, which means that $\frac{1}{2}(y_1 + y_2) \in S_\delta(\phi, x, \epsilon)$. Since $S_\delta(\phi, x, \epsilon)$ scales linearly, we conclude that $\frac{\theta}{2}(y_1 + y_2) \in \theta \cdot S_\delta(\phi, x, \epsilon)$. From these two inclusions, it is clear that

$$\frac{\theta}{2}(y_1 + y_2) \in B[x, \epsilon] \cap \theta \cdot S_\delta(\phi, x, \epsilon) = B[x, \epsilon] \cap S_\delta(\phi, \theta c_x),$$

and equivalently, $B[x, \epsilon] \cap S_\delta(\phi, \theta c_x) \neq \emptyset$. This, however, contradicts the assertion of Lemma 23 since $\theta < 1$.

To summarize, we have established that if the intersection of the sets $B[x, \epsilon]$ and $S_\delta(\phi, x, \epsilon)$ is not a singleton, we can slightly shrink the set $S_\delta(\phi, x, \epsilon)$ such that it still intersects $B[x, \epsilon]$ non-trivially. This is a contradiction in view of Lemma 23.

To prove that y^* defined in (40) is indeed the unique point of intersection, it suffices to show that $y^* \in B[x, \epsilon] \cap S_\delta(\phi, x, \epsilon)$. We observe that:

$$\begin{aligned} \|x - y^*\| &= \left\| x - \frac{\epsilon\phi(f_x) + \delta\mathbf{c}_x x}{\epsilon + \delta\mathbf{c}_x} \right\| = \frac{\epsilon}{\epsilon + \delta\mathbf{c}_x} \|x - \phi(f_x)\| \leq \epsilon, \text{ and} \\ \|y^* - \phi(f_x)\| &= \left\| \frac{\epsilon\phi(f_x) + \delta\mathbf{c}_x x}{\epsilon + \delta\mathbf{c}_x} - \phi(f_x) \right\| = \frac{\delta\mathbf{c}_x}{\epsilon + \delta\mathbf{c}_x} \|x - \phi(f_x)\| \leq \delta\mathbf{c}_x. \end{aligned} \quad (41)$$

These inequalities, along with the fact that $c(f_x) \leq \mathbf{c}_x^p$, imply that $y^* \in B[x, \epsilon] \cap S_\delta(\phi, x, \epsilon)$. This establishes (40).

We proceed to establish the two consequences. to see the first, let us prove that the error constraint is active at the optimal solution (\mathbf{c}_x, f_x) whenever $\|x\| > \epsilon \geq 0$. If $\epsilon = \delta = 0$, then the error constraint is trivially active since the inequality $\|x - \phi(f_x)\| \leq 0$ can only be satisfied with equality. Since $\mathbf{c}_x > 0$ for every $\|x\| > \epsilon$, we have $\epsilon + \delta\mathbf{c}_x > 0$ if at least one of the parameters ϵ and δ is positive, and thus the quantity $y^* = \frac{\epsilon\phi(f_x) + \delta\mathbf{c}_x x}{\epsilon + \delta\mathbf{c}_x}$ is well defined for every $f_x \in F_\delta(\phi, x, \epsilon)$. We know from the previous assertion of the lemma that $y^* \in S_\delta(\phi, x, \epsilon)$. However, since $S_\delta(\phi, x, \epsilon)$ is a convex set that contains 0, we conclude that

$$\theta y^* \in S_\delta(\phi, x, \epsilon) \text{ for every } \theta \in [0, 1].$$

If we suppose that $\|x - \phi(f_x)\| < \epsilon + \delta\mathbf{c}_x$, it is easily verified that $\|x - y^*\| < \epsilon$, and thus $y^* \in B(x, \epsilon)$. As a result, one can find $\rho > 0$ such that $B[y^*, \rho] \subset B[x, \epsilon]$. Since $0 \notin B[x, \epsilon]$, we see at once that $\rho < \|y^*\|$, and conclude that

$$\alpha y^* \in B[y^*, \rho] \subset B[x, \epsilon] \text{ for every } \alpha \text{ such that } \left(1 - \frac{\rho}{\|y^*\|}\right) \leq \alpha \leq 1.$$

These two inclusions together contradict that the sets $B[x, \epsilon]$ and $S_\delta(\phi, x, \epsilon)$ intersect at a unique point.

It remains to prove the final assertion that for every $f_x, g_x \in F_\delta(\phi, x, \epsilon)$, the equality $\phi(f_x) = \phi(g_x)$ holds. Indeed, whenever $\|x\| \leq \epsilon$, we have $C_\delta(\phi, x, \epsilon) = 0$ and $F_\delta(\phi, x, \epsilon) = \{0\}$. This implies that $f_x = g_x = 0$, and thus $\phi(f_x) = \phi(g_x)$. Let us consider the case when $\|x\| > \epsilon$, and suppose that $\phi(f_x) \neq \phi(g_x)$ for some $f_x, g_x \in F_\delta(\phi, x, \epsilon)$. Then it follows that $\frac{1}{2}(f_x + g_x)$ satisfies the error constraint

$$\left\| x - \frac{1}{2}\phi(f_x + g_x) \right\| = \left\| \frac{1}{2}(x - \phi(f_x)) + \frac{1}{2}(x - \phi(g_x)) \right\| \leq \epsilon + \delta\mathbf{c}_x.$$

Moreover, we know that the level sets of c are convex and since $f_x, g_x \in \mathbf{c}_x \cdot V_c$, we have $\frac{1}{2}(f_x + g_x) \in \mathbf{c}_x \cdot V_c$. Therefore, $c(\frac{1}{2}(f_x + g_x)) \leq \mathbf{c}_x^p = C_\delta(\phi, x, \epsilon)$, we conclude that $\frac{1}{2}(f_x + g_x) \in F_\delta(\phi, x, \epsilon)$. However, since $\phi(f_x) \neq \phi(g_x)$, the strict convexity of $\|\cdot\|$ implies that the above error constraint is satisfied strictly. This contradicts our earlier assertion that the error constraint is active for every $f_x \in F_\delta(\phi, x, \epsilon)$. The proof is complete. \blacksquare

Lemma 28 *Let the linear map $\phi : \mathbb{R}^K \longrightarrow \mathbb{H}_n$ and non-negative real numbers ϵ, δ be given, then for every $\lambda \in \mathbb{H}_n$, we have*

$$\|\lambda\|'_\phi = \max_{z \in S_\delta(\phi, 1)} \langle \lambda, z \rangle = \delta \|\lambda\| + \max_{h \in V_c} \langle \lambda, \phi(h) \rangle. \quad (42)$$

Furthermore,

- (i) *If $\delta > 0$, then $\|\lambda\|'_\phi > 0$ for every $\lambda \in \mathbb{H}_n \setminus \{0\}$.*
- (ii) *If $\delta = 0$, and $\lambda \in \mathbb{H}_n \setminus \{0\}$ satisfies $\|\lambda\|'_\phi = 0$, then $\langle \lambda, x \rangle - \epsilon \|\lambda\| \leq 0$ for every $(\phi, \epsilon, 0)$ -feasible vector $x \in \mathbb{H}_n$.*

Proof We recall from the definition (11) that the set $S_\delta(\phi, 1)$ is the image of the linear map: $B[0, \delta] \times V_c \ni (z', h) \longmapsto z' + \phi(h)$. This allows us to write the optimization problem $\max_{z \in S_\delta(\phi, 1)} \langle \lambda, z \rangle$ equivalently as:

$$\max_{h, z'} \langle \lambda, z' + \phi(h) \rangle \quad \text{subject to} \quad h \in V_c, z' \in B[0, \delta].$$

It is easily seen that the above optimization problem is separable into maximization over individual variables, and using the fact that $\max_{z' \in B[0, \delta]} \langle \lambda, z' \rangle = \delta \|\lambda\|$ for every $\lambda \in \mathbb{H}_n$ (42) follows at once. Moreover, since $0 \in V_c$, we have $0 \leq \max_{h \in V_c} \langle \lambda, \phi(h) \rangle$ for every $\lambda \in \mathbb{H}_n$. Applying this inequality in (42), assertion (i) of the lemma follows immediately.

Finally, let $\delta = 0$ and $\lambda \in \mathbb{H}_n \setminus \{0\}$ satisfy $\|\lambda\|'_\phi = 0$. Since $S_0(\phi, 1)$ is an absorbing set to $\text{image}(\phi)$, we conclude from the definition (15) of the dual function that $\langle \lambda, y \rangle \leq 0$ for every $y \in \text{image}(\phi)$. If $x \in \mathbb{H}_n$ is $(\phi, \epsilon, 0)$ -feasible, we know that $B[x, \epsilon] \cap \text{image}(\phi) \neq \emptyset$. Let $y' \in B[x, \epsilon] \cap \text{image}(\phi)$, then

$$\langle \lambda, x \rangle - \epsilon \|\lambda\| = \min_{y \in B[x, \epsilon]} \langle \lambda, y \rangle \leq \langle \lambda, y' \rangle \leq 0.$$

This completes the proof. ■

3.2 Separation of sets $B[x, \epsilon]$ and $S_\delta(\phi, x, \epsilon)$.

We recall that both the sets $B[x, \epsilon]$ and $S_\delta(\phi, x, \epsilon)$ are compact convex subsets that if intersect, then do so at the unique point y^* . Moreover, if $\|x\| > \epsilon$, then we know from the Hahn-Banach separation principle (Boyd and Vandenberghe, 2004, p. 46,50) that there exists a linear functional $\lambda' \in \mathbb{H}_n$ which separates the convex sets $B[x, \epsilon]$ and $S_\delta(\phi, x, \epsilon)$, and supports them at their unique point of intersection y^* . In other words, we have

$$\max_{z \in S_\delta(\phi, x, \epsilon)} \langle \lambda', z \rangle = \langle \lambda', y^* \rangle = \min_{y \in B[x, \epsilon]} \langle \lambda', y \rangle. \quad (43)$$

This equality is central in establishing strong duality and explicitly characterizing the optimal dual variables.

Lemma 29 *Let at least one of ϵ, δ be positive and $x \in \mathbb{H}_n \setminus B[0, \epsilon]$ be (ϕ, ϵ, δ) -feasible. If $0 \neq \lambda' \in \mathbb{H}_n$ satisfies (43), then $\lambda' = \alpha'(x - \phi(f_x))$ for some $\alpha' > 0$ and $f_x \in F_\delta(\phi, x, \epsilon)$. Consequently, (43) is satisfied by $\alpha(x - \phi(f_x))$ for every $\alpha > 0$.*

Proof We recall from the Remark 27 that the sets $B[x, \epsilon]$ and $S_\delta(\phi, x, \epsilon)$ intersect at the unique point y^* , given by

$$y^* = \phi(f_x) + \frac{\delta c_x}{\epsilon + \delta c_x} (x - \phi(f_x)) \mathbb{1}_{]0, +\infty[}(\delta),$$

where $c_x := (C_\delta(\phi, x, \epsilon))^{1/p}$ and $f_x \in F_\delta(\phi, x, \epsilon)$.

- On the one hand, if $\epsilon > 0$ and $\lambda' \neq 0$ satisfies: $\langle \lambda', y^* \rangle = \min_{y \in B[x, \epsilon]} \langle \lambda', y \rangle$, then necessarily $\lambda' = \alpha'(x - y^*)$ for some $\alpha' > 0$.
- On the other hand, if $\delta > 0$, and $\lambda' \neq 0$ satisfies: $\langle \lambda', y^* \rangle = \max_{z \in S_\delta(\phi, x, \epsilon)} \langle \lambda', z \rangle$, then due to the fact that $y^* \in B[\phi(f_x), \delta c_x] \subset S_\delta(\phi, x, \epsilon)$, λ' also satisfies: $\langle \lambda', y^* \rangle = \max_{z \in B[\phi(f_x), \delta c_x]} \langle \lambda', z \rangle$. It follows that: $\lambda' = \alpha''(y^* - \phi(f_x))$ for some $\alpha'' > 0$.

By substituting for y^* and simplifying, we easily deduce that in both the cases $\lambda' = \alpha(x - \phi(f_x))$ for some $\alpha > 0$.

To complete the proof, we argue that if (43) holds true for $\lambda' = \alpha'(x - \phi(f_x))$ with some $\alpha' > 0$, then for any $\alpha > 0$, the inequalities in (43) are preserved by multiplying throughout by the positive quantity $\frac{\alpha}{\alpha'}$. Thus, (43) is satisfied by $\alpha(x - \phi(f_x))$ for every $\alpha > 0$. ■

Lemma 30 *Let the linear map $\phi : \mathbb{R}^K \rightarrow \mathbb{H}_n$ and non-negative real numbers ϵ, δ be given, and $x \in \mathbb{H}_n \setminus B[0, \epsilon]$ be any (ϕ, ϵ, δ) -feasible vector such that $\Lambda_\delta(\phi, x, \epsilon) \neq \emptyset$. Then every $\lambda^* \in \Lambda_\delta(\phi, x, \epsilon)$ satisfies (43).*

Proof We first recall that $S_\delta(\phi, x, \epsilon) = (C_\delta(\phi, x, \epsilon))^{1/p} \cdot S_\delta(\phi, 1)$. Thus, for every $\lambda^* \in \Lambda_\delta(\phi, x, \epsilon)$, the following relations hold:

$$\begin{aligned} \max_{z \in S_\delta(\phi, x, \epsilon)} \langle \lambda^*, z \rangle &= (C_\delta(\phi, x, \epsilon))^{1/p} \max_{z \in S_\delta(\phi, 1)} \langle \lambda^*, z \rangle = (C_\delta(\phi, x, \epsilon))^{1/p}, \text{ and} \\ \min_{y \in B[x, \epsilon]} \langle \lambda^*, y \rangle &= \langle \lambda^*, x \rangle - \epsilon \|\lambda^*\| = (C_\delta(\phi, x, \epsilon))^{1/p}. \end{aligned}$$

In other words, the linear functional $\langle \lambda^*, \cdot \rangle$ separates the sets $B[x, \epsilon]$ and $S_\delta(\phi, x, \epsilon)$. Moreover, both these sets are compact and convex, and we know from Lemma 25 that they intersect at a unique point y^* . Therefore, the linear functional $\langle \lambda^*, \cdot \rangle$ must support both these sets at their intersection point y^* , and (43) follows at once. ■

Proposition 31 *Let the linear map $\phi : \mathbb{R}^K \rightarrow \mathbb{H}_n$ and non-negative real numbers ϵ, δ be given, and $x \in \mathbb{H}_n \setminus B[0, \epsilon]$ be a (ϕ, ϵ, δ) -feasible in the sense of Def. 2. The set $\Lambda_\delta(\phi, x, \epsilon)$ is completely described in the following.*

- (i) If $\delta = 0$, $\epsilon = 0$, then the set $\Lambda_0(\phi, x, 0) \neq \emptyset$, and moreover, $\Lambda_0(\phi, x, 0) \cap \text{image}(\phi) \neq \emptyset$. A vector $\lambda^* \in \Lambda_0(\phi, x, 0)$ if and only if the linear functional $\langle \lambda^*, \cdot \rangle$ supports the set $S_0(\phi, x, 0)$ at x , and satisfies $\|\lambda^*\|'_\phi = 1$.¹⁰
- (ii) If at least one of the following is true
- $\delta > 0$
 - $\delta = 0$ and $\epsilon > 0$ with $B(x, \epsilon) \cap \text{image}(\phi) \neq \emptyset$
- then the set $\Lambda_\delta(\phi, x, \epsilon)$ consists of a unique element λ^* given by

$$\lambda^* = \frac{x - \phi(f_x)}{\|x - \phi(f_x)\|'_\phi} \quad \text{for any } f_x \in F_\delta(\phi, x, \epsilon). \quad (44)$$

- (iii) If $\delta = 0$ and $\epsilon > 0$ such that $B(x, \epsilon) \cap \text{image}(\phi) = \emptyset$, then $\Lambda_\delta(\phi, x, \epsilon) = \emptyset$.

Remark 32 Even though the set $F_\delta(\phi, x, \epsilon)$ may contain multiple elements, λ^* is unique due to the fact that $\phi(f_x)$ is unique.

Proof If $\epsilon = \delta = 0$, then $B[x, \epsilon] = \{x\}$. In view of Lemma 30 we know that $\lambda^* \in \Lambda_0(\phi, x, 0)$ if and only if the linear functional $\langle \lambda^*, \cdot \rangle$ supports the set $S_0(\phi, x, 0)$ at x , and satisfies $\|\lambda^*\|'_\phi = 1$. It remains to be shown that the set $\Lambda_0(\phi, x, 0)$ is non-empty, and we do so by showing that there exists $\lambda_\phi \in \Lambda_0(\phi, x, 0) \cap \text{image}(\phi)$. Since x is $(\phi, 0, 0)$ -feasible, we have $x \in \text{image}(\phi)$. We note from Lemma 23 that $(C_0(\phi, x, 0))^{1/p}$ is the least amount by which the set $S_0(\phi, 1)$ has to be scaled linearly so that it contains x . This implies that x lies on the boundary of the set $S_0(\phi, x, 0)$, i.e., $x \notin \text{relint}(S_0(\phi, x, 0))$ ¹¹. In addition, since $S_0(\phi, x, 0)$ is a convex subset of $\text{image}(\phi)$, we know that there exists $0 \neq \lambda_\phi \in \text{image}(\phi)$ such that the linear functional $\langle \lambda_\phi, \cdot \rangle$ supports the set $S_0(\phi, x, 0)$ at the boundary point x . As result, we obtain:

$$\begin{aligned} \langle \lambda_\phi, x \rangle &= \max_{z \in S_0(\phi, x, 0)} \langle \lambda_\phi, z \rangle = (C_\delta(\phi, x, \epsilon))^{1/p} \max_{z \in S_0(\phi, 1)} \langle \lambda_\phi, z \rangle \\ &= (C_\delta(\phi, x, \epsilon))^{1/p} \|\lambda_\phi\|'_\phi. \end{aligned}$$

Since $S_0(\phi, 1)$ is an absorbing set to $\text{image}(\phi)$ we have $0 \in \text{relint}(S_0(\phi, 1))$ and therefore, $0 < \|\lambda_\phi\|'_\phi$. Thus, defining $\lambda^* := (1/\|\lambda_\phi\|'_\phi)\lambda_\phi$ it readily follows that $\lambda^* \in \Lambda_0(\phi, x, 0)$. This establishes the assertion (i) of the proposition.

If either $\epsilon > 0$ or $\delta > 0$, on the one hand we know from Lemma 29 that $(x - \phi(f_x))$ satisfies

$$\begin{aligned} \langle x - \phi(f_x), x \rangle - \epsilon \|x - \phi(f_x)\| &= \max_{z \in S_\delta(\phi, x, \epsilon)} \langle x - \phi(f_x), z \rangle \\ &= (C_\delta(\phi, x, \epsilon))^{1/p} \|x - \phi(f_x)\|'_\phi. \end{aligned}$$

We immediately see that if $\|x - \phi(f_x)\|'_\phi > 0$, then $\frac{x - \phi(f_x)}{\|x - \phi(f_x)\|'_\phi} \in \Lambda_\delta(\phi, x, \epsilon)$. On the other hand, if $\lambda^* \in \Lambda_\delta(\phi, x, \epsilon)$ then Lemma 30 implies that λ^* must satisfy (43), and from Lemma

10. If $\text{image}(\phi)$ is a proper subspace of \mathbb{H}_n , then every λ in the orthogonal complement of $\text{image}(\phi)$ supports the set $S_0(\phi, 1)$ at every point, and in particular at x . However, such a λ doesn't satisfy the condition $\langle \lambda, x \rangle - \epsilon \|\lambda\| = (C_\delta(\phi, x, \epsilon))^{1/p}$.

11. For a given set \mathcal{S} , its *relative interior* $\text{relint}(\mathcal{S})$ is defined as the interior of the set w.r.t. the subspace topology induced on the smallest affine subspace containing the set \mathcal{S} .

29 we infer that $\lambda^* = \alpha(x - \phi(f_x))$ for some $\alpha > 0$. From Definition 6 it immediately implies that if $\alpha(x - \phi(f_x)) \in \Lambda_\delta(\phi, x, \epsilon)$, then $\|x - \phi(f_x)\|'_\phi > 0$ and $\alpha = \frac{1}{\|x - \phi(f_x)\|'_\phi}$. Thus the set $\Lambda_\delta(\phi, x, \epsilon)$ is non-empty, and is the singleton $\left\{ \frac{x - \phi(f_x)}{\|x - \phi(f_x)\|'_\phi} \right\}$ if and only if $\|x - \phi(f_x)\|'_\phi > 0$.

We complete the proof by showing that $\|x - \phi(f_x)\|'_\phi = 0$ if and only if $\delta = 0$ and $B(x, \epsilon) \cap \text{image}(\phi) = \emptyset$. On the one hand, if $\delta = 0$ and $B(x, \epsilon) \cap \text{image}(\phi) = \emptyset$, then we have

$$\|x - \pi_\phi(x)\| = \min_{z \in \text{image}(\phi)} \|x - z\| \geq \epsilon.$$

However, from Lemma 25 we know that $\|x - \phi(f_x)\| = \epsilon$, and since $\phi(f_x) \in \text{image}(\phi)$, we deduce that $\pi_\phi(x) = \phi(f_x)$.¹² Due to orthogonality of projection, $\langle x - \phi(f_x), z \rangle = 0$ for all $z \in \text{image}(\phi)$. Since $S_0(\phi, 1) \subset \text{image}(\phi)$, we obtain

$$\|x - \phi(f_x)\|'_\phi = \max_{z \in S_0(\phi, 1)} \langle x - \phi(f_x), z \rangle = 0$$

On the other hand, if $\|x - \phi(f_x)\|'_\phi = 0$, Lemma 28(i) implies that $\delta = 0$. Moreover, since $S_0(\phi, 1)$ is an absorbing set to $\text{image}(\phi)$, we conclude from the definition (15) of the dual function that $\langle x - \phi(f_x), z \rangle = 0$ for all $z \in \text{image}(\phi)$. Furthermore, since $\phi(f_x) \in \text{image}(\phi)$ it implies from the orthogonality principle that $\pi_\phi(x) = \phi(f_x)$. Consequently,

$$\min_{z \in \text{image}(\phi)} \|x - z\| = \|x - \pi_\phi(x)\| = \|x - \phi(f_x)\| = \epsilon.$$

In other words, we have $B(x, \epsilon) \cap \text{image}(\phi) = \emptyset$. The proof is now complete. \blacksquare

Lemma 33 *Let the linear map $\phi : \mathbb{R}^K \rightarrow \mathbb{H}_n$ and non-negative real numbers ϵ, δ be given, and $x \in \mathbb{H}_n \setminus B[0, \epsilon]$ be (ϕ, ϵ, δ) -feasible such that $\Lambda_\delta(\phi, x, \epsilon) \neq \emptyset$. Then for every $\lambda^* \in \Lambda_\delta(\phi, x, \epsilon)$ and $h^* \in \frac{1}{(C_\delta(\phi, x, \epsilon))^{1/p}} \cdot F_\delta(\phi, x, \epsilon)$, we have*

$$\langle \lambda^*, \phi(h^*) \rangle = \max_{h \in V_c} \langle \lambda^*, \phi(h) \rangle = 1 - \delta \|\lambda^*\|. \quad (45)$$

Proof Applying (42) directly to $\lambda^* \in \Lambda_\delta(\phi, x, \epsilon)$ gives us

$$\max_{h \in V_c} \langle \lambda^*, \phi(h) \rangle = -\delta \|\lambda^*\| + \|\lambda^*\|'_\phi = 1 - \delta \|\lambda^*\|. \quad (46)$$

By denoting $\mathbf{c}_x = (C_\delta(\phi, x, \epsilon))^{1/p}$, we know from (43) and Lemma 30 that

$$\langle \lambda^*, y^* \rangle = \max_{z \in S_\delta(\phi, x, \epsilon)} \langle \lambda^*, z \rangle = \mathbf{c}_x \max_{z \in S_\delta(\phi, 1)} \langle \lambda^*, z \rangle = \mathbf{c}_x \|\lambda^*\|'_\phi = \mathbf{c}_x.$$

On substituting for y^* by considering $f_x = \mathbf{c}_x h^*$ in Remark 27, we get

$$\mathbf{c}_x = \langle \lambda^*, y^* \rangle = \mathbf{c}_x \langle \lambda^*, \phi(h^*) \rangle + \frac{\delta \mathbf{c}_x}{\epsilon + \delta \mathbf{c}_x} \langle \lambda^*, (x - \mathbf{c}_x \phi(h^*)) \rangle \mathbb{1}_{]0, +\infty[}(\delta). \quad (47)$$

12. $\pi_\phi : \mathbb{H}_n \rightarrow \text{image}(\phi)$ is the orthogonal projection operator onto $\text{image}(\phi)$.

Whenever $\delta > 0$ we know from Proposition 31 that λ^* and $(x - \mathbf{c}_x \phi(h^*))$ are co-linear. Thus, we obtain that:

$$\langle \lambda^*, (x - \mathbf{c}_x \phi(h^*)) \rangle = \|\lambda^*\| \|x - \mathbf{c}_x \phi(h^*)\| = (\epsilon + \delta \mathbf{c}_x) \|\lambda^*\|,$$

where the last equality follows from assertion (i) of Lemma 25 since $\mathbf{c}_x h^* \in F_\delta(\phi, x, \epsilon)$. Note that $\mathbf{c}_x > 0$ since $\|x\| > \epsilon$. Therefore, cancelling \mathbf{c}_x throughout in (47) and simplifying for $\langle \lambda^*, \phi(h^*) \rangle$ yields

$$\langle \lambda^*, \phi(h^*) \rangle = 1 - \left(\delta \|\lambda^*\| \mathbb{1}_{]0, +\infty[}(\delta) \right) = 1 - \delta \|\lambda^*\|, \quad (48)$$

(45) follows at once from (46) and (48). ■

Proof [Lemma 5] If x is not (ϕ, ϵ, δ) -feasible, then we know that $\delta = 0$ and $B[x, \epsilon] \cap \text{image}(\phi) = \emptyset$. Consequently, $\|y\|_\phi = +\infty$ for all $y \in B[x, \epsilon]$. Therefore, the assertion holds since $C_\delta(\phi, x, \epsilon) = +\infty$.

If x is (ϕ, ϵ, δ) -feasible, then from Lemma 25, we know that the sets $B[x, \epsilon]$ and $S_\delta(\phi, x, \epsilon)$ intersect at a unique point y^* . Thus we have

$$\min_{y \in B[x, \epsilon]} \|y\|_\phi \leq \|y^*\|_\phi \leq (C_\delta(\phi, x, \epsilon))^{1/p},$$

where the first inequality follows from the fact that $y^* \in B[x, \epsilon]$ and the second one follows from $y^* \in (C_\delta(\phi, x, \epsilon))^{1/p} \cdot S_\delta(\phi, 1)$ and the definition (13) of the gauge function $\|\cdot\|_\phi$.

On the one hand, for $y \in B[x, \epsilon]$ such that $\|y\|_\phi = +\infty$, the inequality $(C_\delta(\phi, x, \epsilon))^{1/p} \leq \|y\|_\phi$ holds trivially. On the other hand, for $y \in B[x, \epsilon]$ such that $\|y\|_\phi < +\infty$, we know from the definition (13) that $y \in S_\delta(\phi, \|y\|_\phi)$. Thus, $B[x, \epsilon] \cap S_\delta(\phi, \|y\|_\phi) \neq \emptyset$, and in view of Lemma 23, we get $(C_\delta(\phi, x, \epsilon))^{1/p} \leq \|y\|_\phi$. Combining the two facts, we conclude

$$(C_\delta(\phi, x, \epsilon))^{1/p} \leq \min_{y \in B[x, \epsilon]} \|y\|_\phi.$$

Collecting the two inequalities, (14) follows at once. ■

Remark 34 The proof of the lemma also implies that $\|y^*\|_\phi = (C_\delta(\phi, x, \epsilon))^{1/p}$, and therefore, y^* is a minimizer in the problem (14). Furthermore, if $y' \neq y^*$ is also a minimizer, then we have $\|y'\|_\phi = (C_\delta(\phi, x, \epsilon))^{1/p}$ and $y' \in B[x, \epsilon]$. Then it follows that $y' \in S_\delta(\phi, \|y'\|_\phi) = S_\delta(\phi, x, \epsilon)$, and thus $y' \in B[x, \epsilon] \cap S_\delta(\phi, x, \epsilon)$. From Lemma 25, we then have $y' = y^*$. Which is a contradiction. Thus,

$$y^* = \operatorname{argmin}_{y \in B[x, \epsilon]} \|y\|_\phi.$$

Proof [Theorem 7] Combining (14) and (16), we obtain

$$\begin{aligned}
 (C_\delta(\phi, x, \epsilon))^{1/p} &= \min_{y \in B[x, \epsilon]} \sup_{\|\lambda\|'_\phi \leq 1} \langle \lambda, y \rangle \\
 &\geq \sup_{\|\lambda\|'_\phi \leq 1} \min_{y \in B[x, \epsilon]} \langle \lambda, y \rangle \\
 &\geq \begin{cases} \sup_{\lambda} & \langle \lambda, x \rangle - \epsilon \|\lambda\| \\ \text{subject to} & \|\lambda\|'_\phi \leq 1. \end{cases} \tag{49}
 \end{aligned}$$

Therefore, $(C_\delta(\phi, x, \epsilon))^{1/p}$ is an upper bound to the optimal value of (17). We shall establish the proposition by considering all the possible cases and showing that the upper bound is indeed the supremum.

Case 1: When x is not (ϕ, ϵ, δ) -feasible. We know that this happens only if $\delta = 0$ and $B[x, \epsilon] \cap \text{image}(\phi) = \emptyset$. Denoting $\pi_\phi(x)$ to be the orthogonal projection of x onto $\text{image}(\phi)$, we have $\langle x - \pi_\phi(x), z \rangle = 0$ for every $z \in \text{image}(\phi)$.

Since $\delta = 0$ we have $S_0(\phi, 1) \subset \text{image}(\phi)$. Thus, for every $\alpha \geq 0$, letting $\lambda'_\alpha := \alpha(x - \pi_\phi(x))$ we see that $\langle \lambda'_\alpha, z \rangle = 0$ for every $z \in S_0(\phi, 1)$. In other words, we have $\|\lambda'_\alpha\|'_\phi = 0$, and therefore, λ'_α is a feasible point in (17) for every $\alpha \geq 0$. Moreover, since $B[x, \epsilon] \cap \text{image}(\phi) = \emptyset$ we see that $\|x - \pi_\phi(x)\| \geq \epsilon + \rho$ for some $\rho > 0$. Therefore, the objective function of (17) evaluated at λ_α satisfies

$$\begin{aligned}
 \langle \lambda'_\alpha, x \rangle - \epsilon \|\lambda'_\alpha\| &= \alpha \left(\langle x - \pi_\phi(x), x \rangle - \epsilon \|x - \pi_\phi(x)\| \right) \\
 &= \alpha \left(\|x - \pi_\phi(x)\|^2 + \langle x - \pi_\phi(x), \pi_\phi(x) \rangle - \epsilon \|x - \pi_\phi(x)\| \right) \\
 &= \alpha \|x - \pi_\phi(x)\| \left(\|x - \pi_\phi(x)\| - \epsilon \right) \\
 &\geq \alpha(\epsilon + \rho)\rho.
 \end{aligned}$$

By considering arbitrarily large value of α , we observe that the objective function in (17) attains arbitrarily large values for λ'_α , i.e., the supremum is $+\infty$.

Case 2: When $0 \leq \|x\| \leq \epsilon$. We know that the optimal cost $C_\delta(\phi, x, \epsilon)$ is equal to zero, and we shall conclude that so is the value of the supremum in (17). For $\lambda^* = 0$, we have $\|\lambda^*\|'_\phi = 0$ (thus, λ^* is feasible in (17)) and $\langle \lambda^*, x \rangle - \epsilon \|\lambda^*\| = 0$. Therefore, in view of (49), we know that the objective function in (17) achieves the value of its upper bound at λ^* . Thus, the value of the supremum is 0, and $\lambda^* = 0$ is an optimal solution.¹³

Case 3: When x is a (ϕ, ϵ, δ) -feasible, and $\|x\| > \epsilon$ with $\Lambda_\delta(\phi, x, \epsilon) \neq \emptyset$. We know that there exists a $\lambda^* \in \Lambda_\delta(\phi, x, \epsilon)$ and the following two conditions hold simultaneously:

$$\begin{aligned}
 \|\lambda^*\|'_\phi &= 1, \text{ and} \\
 \langle \lambda^*, x \rangle - \epsilon \|\lambda^*\| &= (C_\delta(\phi, x, \epsilon))^{1/p}.
 \end{aligned}$$

13. It is to be noted that whenever $\|x\| = \epsilon$, there could be non-zero optimal solutions, for e.g., $\lambda^* = \alpha x$ for every $\alpha \geq 0$.

The first equality implies that λ^* is a feasible point to (17), and the latter, in conjunction with (49) implies that the upper bound of $(C_\delta(\phi, x, \epsilon))^{1/p}$ is achieved at λ^* . Thus, $(C_\delta(\phi, x, \epsilon))^{1/p}$ is indeed the optimum value of (17), and that every $\lambda^* \in \Lambda_\delta(\phi, x, \epsilon)$ is an optimal solution to (17).

Conversely, if λ^* is an optimal solution to (17), then readily we get $\langle \lambda^*, x \rangle - \epsilon \|\lambda^*\| = (C_\delta(\phi, x, \epsilon))^{1/p}$. It suffices to show that $\|\lambda^*\|'_\phi = 1$. Since $(C_\delta(\phi, x, \epsilon))^{1/p} > 0$, we have $\langle \lambda^*, x \rangle - \epsilon \|\lambda^*\| > 0$. Therefore, from the assertions (i) and (ii) of Lemma 28, we conclude that $\|\lambda^*\|'_\phi > 0$. Moreover, if $\|\lambda^*\|'_\phi < 1$, then $\lambda' := \frac{1}{\|\lambda^*\|'_\phi} \lambda^*$ is also a feasible point to (17). However, the objective function of (17) evaluated at λ' satisfies

$$\langle \lambda', x \rangle - \epsilon \|\lambda'\| = \frac{1}{\|\lambda^*\|'_\phi} (\langle \lambda^*, x \rangle - \epsilon \|\lambda^*\|) > (C_\delta(\phi, x, \epsilon))^{1/p},$$

which is a contradiction. Therefore, it follows at once that $\lambda^* \in \Lambda_\delta(\phi, x, \epsilon)$.

Case 4: When x is a (ϕ, ϵ, δ) -feasible vector and $\|x\| > \epsilon$ with $\Lambda_\delta(\phi, x, \epsilon) = \emptyset$. We know from Proposition 31 that this happens only if $\delta = 0$ and $B(x, \epsilon) \cap \text{image}(\phi) = \emptyset$. Since x is $(\phi, \epsilon, 0)$ -feasible, we also know that $B[x, \epsilon] \cap \text{image}(\phi) \neq \emptyset$. Thus the intersection must happen at the boundary of the set $B[x, \epsilon]$. Due to the strict convex geometry of $B[x, \epsilon]$, the sets $B[x, \epsilon]$ and $\text{image}(\phi)$ can only intersect at a unique point, observe that this is also the point that is closest to x among all points in $\text{image}(\phi)$. Thus we conclude that $B[x, \epsilon] \cap \text{image}(\phi) = \pi_\phi(x)$ - the orthogonal projection of x onto $\text{image}(\phi)$. Since no point other than $\pi_\phi(x)$ in $B[x, \epsilon]$ intersects with $\text{image}(\phi)$, the LIP (8) reduces to the following:

$$\begin{cases} \text{minimize} & c(f) \\ f \in \mathbb{R}^K & \\ \text{subject to} & \phi(f) = \pi_\phi(x), \end{cases}$$

which simply is another LIP with parameters $\pi_\phi(x)$, ϕ and $\epsilon = \delta = 0$. Since, $\pi_\phi(x) \in \text{image}(\phi)$, $\pi_\phi(x)$ is $(\phi, 0, 0)$ -feasible. Therefore, $C_0(\phi, x, \epsilon) = C_0(\phi, \pi_\phi(x), 0)$ and $F_0(\phi, x, \epsilon) = F_0(\phi, \pi_\phi(x), 0)$. In addition, from the Proposition 31 it follows that the set $\Lambda_0(\phi, \pi_\phi(x), 0)$ is non-empty, and in particular, there exists $\lambda' \in \text{image}(\phi) \cap \Lambda_0(\phi, \pi_\phi(x), 0)$ such that the following two conditions hold simultaneously.

$$\langle \lambda', \pi_\phi(x) \rangle = (C_0(\phi, \pi_\phi(x), 0))^{1/p} = (C_0(\phi, x, \epsilon))^{1/p} \text{ and } \|\lambda'\|'_\phi = 1$$

Using the above facts, we shall first establish that the value $(C_\delta(\phi, x, \epsilon))^{1/p}$ is not just an upper bound but is indeed the supremum in (17).

For every $\alpha \geq 0$ let $\lambda(\alpha) := \lambda' + \alpha(x - \pi_\phi(x))$. Since the linear functional $\langle x - \pi_\phi(x), \cdot \rangle$ vanishes on $\text{image}(\phi)$, for every $z \in \text{image}(\phi)$ we have

$$\begin{aligned} \langle \lambda(\alpha), z \rangle &= \langle \lambda', z \rangle + \alpha \langle x - \pi_\phi(x), z \rangle = \langle \lambda', z \rangle, \text{ and therefore,} \\ \|\lambda(\alpha)\|'_\phi &= \max_{z \in S_0(\phi, 1)} \langle \lambda(\alpha), z \rangle = \max_{z \in S_0(\phi, 1)} \langle \lambda', z \rangle = \|\lambda'\|'_\phi = 1. \end{aligned}$$

Thus, $\lambda(\alpha)$ is a feasible point to (17), and the objective function evaluated at $\lambda(\alpha)$ satisfies:

$$\begin{aligned} \langle \lambda(\alpha), x \rangle - \epsilon \|\lambda(\alpha)\| &= \langle \lambda(\alpha), \pi_\phi(x) \rangle + \langle \lambda(\alpha), x - \pi_\phi(x) \rangle - \epsilon \|\lambda(\alpha)\| \\ &= \langle \lambda', \pi_\phi(x) \rangle + \alpha \|x - \pi_\phi(x)\|^2 - \epsilon \sqrt{\|\lambda'\|^2 + \alpha^2 \|x - \pi_\phi(x)\|^2} \\ &= (C_0(\phi, x, \epsilon))^{1/p} + \epsilon \left(\alpha \epsilon - \sqrt{\|\lambda'\|^2 + \alpha^2 \epsilon^2} \right). \end{aligned}$$

Since $\lambda(\alpha)$ is feasible in (17) for every $\alpha \geq 0$, the supremum in (17) is sandwiched between $\sup_{\alpha \geq 0} \langle \lambda(\alpha), x \rangle - \epsilon \|\lambda(\alpha)\|$ and the optimal cost $(C_0(\phi, x, \epsilon))^{1/p}$. However, we see that:

$$\sup_{\alpha \geq 0} \langle \lambda(\alpha), x \rangle - \epsilon \|\lambda(\alpha)\| \geq \lim_{\alpha \rightarrow +\infty} \langle \lambda(\alpha), x \rangle - \epsilon \|\lambda(\alpha)\| = (C_0(\phi, x, \epsilon))^{1/p}.^{14}$$

This implies that the supremum in (17) is indeed equal to $(C_0(\phi, x, \epsilon))^{1/p}$.

Now that we know the value of the supremum, it suffices to establish that (17) does not admit an optimal solution in this case. If there were any λ' that is an optimal solution to (17), then from the arguments provided in the proof of necessary implication for case 3, it follows that $\lambda' \in \Lambda_0(\phi, x, \epsilon)$. This contradicts the premise $\Lambda_0(\phi, x, \epsilon) = \emptyset$. Therefore, (17) admits no solution whenever $\Lambda_0(\phi, x, \epsilon) = \emptyset$. \blacksquare

Lemma 35 *Let the linear map $\phi : \mathbb{R}^K \rightarrow \mathbb{H}_n$, non-negative real numbers ϵ, δ and $x \in \mathbb{H}_n \setminus B[0, \epsilon]$ be given. For any $h \in V_c$, consider the optimization problem*

$$\begin{cases} \sup_{\lambda} & \langle \lambda, x \rangle - \epsilon \|\lambda\| \\ \text{subject to} & \begin{cases} \langle \lambda, x \rangle - \epsilon \|\lambda\| > 0, \\ \langle \lambda, \phi(h) \rangle + \delta \|\lambda\| \leq 1. \end{cases} \end{cases} \quad (50)$$

(i) *The optimal value of (50) is equal to*

$$\eta_h := \inf \{ \theta \geq 0 : B[x, \epsilon] \cap B[\phi(\theta h), \theta \delta] \neq \emptyset \}. \quad (51)$$

- (ii) $\eta_h \geq (C_\delta(\phi, x, \epsilon))^{1/p}$ and equality holds (whenever x is (ϕ, ϵ, δ) -feasible) if and only if $h \in \frac{1}{(C_\delta(\phi, x, \epsilon))^{1/p}} F_\delta(\phi, x, \epsilon)$.
- (iii) $\eta_h = +\infty$ if and only if there exists a $\lambda' \in \mathbb{H}_n$ that simultaneously satisfies the conditions

- $\langle \lambda', x \rangle - \epsilon \|\lambda'\| > 0$
- $\langle \lambda', \phi(h) \rangle + \delta \|\lambda'\| \leq 0$.

14. For any $b > 0$, we see that

$$\lim_{\alpha \rightarrow +\infty} (\alpha - \sqrt{b + \alpha^2}) = \lim_{\alpha \rightarrow +\infty} \frac{(\alpha - \sqrt{b + \alpha^2})(\alpha + \sqrt{b + \alpha^2})}{(\alpha + \sqrt{b + \alpha^2})} = \lim_{\alpha \rightarrow +\infty} \frac{-b}{\alpha + \sqrt{\alpha^2 + b}} = 0.$$

Proof Let the map $L(\eta, h) : [0, +\infty[\times V_c \longrightarrow [0, +\infty]$ be defined by

$$L(\eta, h) := \begin{cases} \sup_{\lambda} & \left(\langle \lambda, x \rangle - \epsilon \|\lambda\| \right) - \eta \left(\langle \lambda, \phi(h) \rangle + \delta \|\lambda\| \right) \\ \text{subject to} & \langle \lambda, x \rangle - \epsilon \|\lambda\| > 0. \end{cases} \quad (52)$$

For every $\eta \geq 0$, let us define the set $S'(\eta) := \bigcup_{\theta \in [0, \eta]} B[\phi(\theta h), \delta \theta]$. It is easily verified that $S'(\eta)$ is a convex-compact subset of \mathbb{H}_n and monotonic, i.e., $S'(\eta) \subset S'(\eta')$ for every $\eta \leq \eta'$.

For every $h \in V_c$ and $\theta \geq 0$, we observe that $B[\phi(\theta h), (\theta \delta)] = \theta \cdot B[\phi(h), \delta]$. Since the sets $B[x, \epsilon]$ and $B[\phi(h), \delta]$ are compact, the minimization over $\theta \geq 0$ in (51) is achieved whenever it is finite. Therefore, we have $B[x, \epsilon] \cap B[\phi(\eta h), (\eta \delta)] \neq \emptyset$. On the one hand, for $0 \leq \eta < \eta_h \leq +\infty$, we know that the convex sets $B[x, \epsilon]$ and $S'(\eta)$ do not intersect. Therefore, there exists a non-zero $\lambda' \in \mathbb{H}_n$ such that the linear functional $\langle \lambda', \cdot \rangle$ separates them. In other words, we have

$$\min_{y \in B[x, \epsilon]} \langle \lambda', y \rangle > \max_{z \in S'(\eta)} \langle \lambda', z \rangle.$$

Observing the following equalities

$$\begin{aligned} \min_{y \in B[x, \epsilon]} \langle \lambda', y \rangle &= \langle \lambda', x \rangle - \epsilon \|\lambda'\|, \quad \text{and} \\ \max_{z \in S'(\eta)} \langle \lambda', z \rangle &= \max \left\{ 0, \max_{z \in B[\phi(\eta h), \delta \eta]} \langle \lambda', z \rangle \right\} \\ &= \max \left\{ 0, \eta \left(\langle \lambda', \phi(h) \rangle + \delta \|\lambda'\| \right) \right\}, \end{aligned}$$

we get

$$\langle \lambda', x \rangle - \epsilon \|\lambda'\| > \max \left\{ 0, \eta \left(\langle \lambda', \phi(h) \rangle + \delta \|\lambda'\| \right) \right\}. \quad (53)$$

It follows at once that for every $\alpha \geq 0$, $\lambda'_\alpha := \alpha \lambda'$ is a feasible point in (52), and thus,

$$\begin{aligned} L(\eta, h) &\geq \sup_{\alpha \geq 0} \left\{ \left(\langle \lambda'_\alpha, x \rangle - \epsilon \|\lambda'_\alpha\| \right) - \eta \left(\langle \lambda'_\alpha, \phi(h) \rangle + \delta \|\lambda'_\alpha\| \right) \right\} \\ &= \left(\left(\langle \lambda', x \rangle - \epsilon \|\lambda'\| \right) - \eta \left(\langle \lambda', \phi(h) \rangle + \delta \|\lambda'\| \right) \right) \left(\sup_{\alpha \geq 0} \alpha \right) \\ &= +\infty. \end{aligned}$$

On the other hand, for $\eta_h \leq \eta < +\infty$, we know that $B[x, \epsilon] \cap S'(\eta) \neq \emptyset$. Therefore, for every $\lambda \in \mathbb{H}_n$, we have

$$\langle \lambda, x \rangle - \epsilon \|\lambda\| = \min_{y \in B[x, \epsilon]} \langle \lambda, y \rangle \leq \max_{z \in S'(\eta)} \langle \lambda, z \rangle = \max \left\{ 0, \eta \left(\langle \lambda, \phi(h) \rangle + \delta \|\lambda\| \right) \right\}.$$

Therefore, for every λ satisfying $\langle \lambda, x \rangle - \epsilon \|\lambda\| > 0$, we obtain $\langle \lambda, \phi(h) \rangle + \delta \|\lambda\| > 0$, and

$$\langle \lambda, x \rangle - \epsilon \|\lambda\| \leq \eta \left(\langle \lambda, \phi(h) \rangle + \delta \|\lambda\| \right).$$

By taking the supremum over all λ , we obtain $L(\eta, h) \leq 0$. However, by picking any λ such that $\langle \lambda, x \rangle - \epsilon \|\lambda\| > 0$, and defining $\lambda_\alpha := \alpha\lambda$ for every $\alpha > 0$, we immediately see that $\langle \lambda_\alpha, x \rangle - \epsilon \|\lambda_\alpha\| > 0$ and

$$0 = \lim_{\alpha \rightarrow 0} \left(\langle \lambda_\alpha, x \rangle - \epsilon \|\lambda_\alpha\| \right) - \eta \left(\langle \lambda_\alpha, \phi(h) \rangle + \delta \|\lambda_\alpha\| \right).$$

Therefore, $L(\eta, h) = 0$. Summarizing, we have:¹⁵

$$L(\eta, h) = \begin{cases} +\infty & \text{if } 0 \leq \eta < \eta_h \\ 0 & \text{if } \eta_h \leq \eta < \infty. \end{cases}$$

Let us consider the Lagrange-dual of (50) (Boyd and Vandenberghe, 2004, Section 5.2), which is written in the following inf-sup formulation.

$$\begin{cases} \inf_{\eta \geq 0} \sup_{\lambda} & \langle \lambda, x \rangle - \epsilon \|\lambda\| - \eta (\delta \|\lambda\| + \langle \lambda, \phi(h) \rangle - 1) \\ \text{subject to} & \langle \lambda, x \rangle - \epsilon \|\lambda\| > 0. \end{cases} \quad (54)$$

Solving for the supremum over λ , the inf-sup problem (54) reduces to $\inf_{\eta \geq 0} \eta + L(\eta, h)$. It is immediate that the optimal value of the inf-sup problem (54) is equal to η_h .

We observe that the optimization problem (50), is a convex program. Moreover, since $\|x\| > \epsilon$, we see that $\lambda' := \alpha x$ is a strictly feasible point in (50) for every $0 < \alpha < \frac{1}{\langle x, \phi(h) \rangle + \delta \|x\|}$ (and for every $\alpha > 0$ if $\langle x, \phi(h) \rangle + \delta \|x\| = 0$). Therefore, strong duality holds for the convex problem (50), and the optimal value of (50) is indeed equal to η_h . This establishes the assertion (i) of the lemma.

Since $B[\phi(h), \delta] \subset S_\delta(\phi, 1)$ for any $h \in V_c$, we see that

$$B[\phi(\eta h), \delta \eta] = \eta \cdot B[\phi(h), \delta] \subset \eta \cdot S_\delta(\phi, 1) \subset S_\delta(\phi, \eta) \text{ for every } \eta > 0. \quad (55)$$

If $\eta_h = +\infty$, assertion (ii) of the lemma holds true trivially. If $\eta_h \in [0, +\infty[$, the fact that $B[x, \epsilon] \cap B[\phi(\eta_h h), (\eta_h \delta)] \neq \emptyset$ together with (55) (for $\eta = \eta_h$) immediately gives us $B[x, \epsilon] \cap S_\delta(\phi, \eta_h) \neq \emptyset$. Thus, in view of Lemma 23, we have $\eta_h \geq (C_\delta(\phi, x, \epsilon))^{1/p}$. Now, if $\eta_{h'} = c_x := (C_\delta(\phi, x, \epsilon))^{1/p}$ for some $h' \in V_c$, then by considering $f' = c_x h'$, we first see that $c(f') \leq C_\delta(\phi, x, \epsilon)$. Moreover, since $B[x, \epsilon] \cap B[\phi(c_x h'), (c_x \delta)] \neq \emptyset$, we also have $\|x - \phi(c_x h')\| \leq \epsilon + \delta c_x$. Thus, $c_x h' \in F_\delta(\phi, x, \epsilon)$. Conversely, for every $f' \in F_\delta(\phi, x, \epsilon)$, we conclude from (41) that $B[x, \epsilon] \cap B[\phi(f'), \delta c_x] \neq \emptyset$. Therefore, $\eta_{(c_x^{-1} f')} \leq (C_\delta(\phi, x, \epsilon))^{1/p}$, but since $\eta_h \geq (C_\delta(\phi, x, \epsilon))^{1/p}$ for every $h \in V_c$, we conclude that $\eta_{(c_x^{-1} f')} = (C_\delta(\phi, x, \epsilon))^{1/p}$. This establishes assertion (ii) of the lemma.

If there exists a $\lambda' \in \mathbb{H}_n$ such that the conditions $\langle \lambda', x \rangle - \epsilon \|\lambda'\| > 0$ and $\langle \lambda', \phi(h) \rangle + \delta \|\lambda'\| \leq 0$ hold simultaneously. Then for every $\alpha > 0$, $\lambda_\alpha := \alpha \lambda'$ is a feasible point in (50). Therefore, we have

$$\eta_h \geq \sup_{\alpha > 0} \langle \lambda_\alpha, x \rangle - \epsilon \|\lambda_\alpha\| = +\infty.$$

¹⁵ It is to be noted that if $\eta_h = +\infty$, then $L(\eta, h) = +\infty$ for every $\eta \in [0, +\infty[$.

Conversely let $\eta_h = +\infty$, then we know that the compact-convex set $B[x, \epsilon]$ does not intersect with the closed convex-cone $S' := \bigcup_{\theta \in [0, +\infty[} B[\phi(\theta h), \delta\theta]$. Since one of the sets involved is

compact, there exists a $\lambda' \in \mathbb{H}_n$ such that the linear functional $\langle \lambda', \cdot \rangle$ separates these sets *strictly*. Thus, we have

$$\max_{z \in S'} \langle \lambda', z \rangle < \min_{y \in B[x, \epsilon]} \langle \lambda', y \rangle = \langle \lambda', x \rangle - \epsilon \|\lambda'\|.$$

We note that the quantity $\langle \lambda', x \rangle - \epsilon \|\lambda'\|$ is a minimum of a linear functional over a compact set, and thus finite. On the contrary, $\max_{z \in S'} \langle \lambda', z \rangle$ is a maximum of the linear functional λ' over the cone S' . Therefore, it can be either 0 or $+\infty$. However, since $\langle \lambda', x \rangle - \epsilon \|\lambda'\|$ is an upper bound to this maximum, we have $0 = \max_{z \in S'} \langle \lambda', z \rangle$. Therefore, we get $\langle \lambda', x \rangle - \epsilon \|\lambda'\| > 0$, and since $B[\phi(h), \delta] \subset S'$ we also have $\langle \lambda', \phi(h) \rangle + \delta \|\lambda'\| \leq 0$. This completes the proof. \blacksquare

Lemma 36 *Let the linear map ϕ , real numbers $\epsilon, \delta \geq 0$, $q \in]0, 1[$, $r > 0$ and $x \in \mathbb{H}_n \setminus B[x, \epsilon]$ be given. For every $h \in V_c$, let us consider the following optimization problem:*

$$\begin{cases} \sup_{\lambda} & r \left(\langle \lambda, x \rangle - \epsilon \|\lambda\| \right)^q - \left(\delta \|\lambda\| + \langle \lambda, \phi(h) \rangle \right) \\ \text{subject to} & \langle \lambda, x \rangle - \epsilon \|\lambda\| > 0. \end{cases} \quad (56)$$

- (i) *If $q \in]0, 1[$, the optimal value of (56) is $s(r, q) \eta_h^{\frac{q}{1-q}}$, where η_h is as defined in (51) and $s(r, q)$ is some constant.¹⁶*
- (ii) *If $q = 1$, the optimal value of (56) is finite and equal to 0 if and only if $\eta_h \leq \frac{1}{r}$.*

Proof We begin by considering the case when $\eta_h < +\infty$. From the assertion (iii) of Lemma 35, it follows that $\langle \lambda, \phi(h) \rangle + \delta \|\lambda\| > 0$ for every $\lambda \in \mathbb{H}_n$ satisfying $\langle \lambda, x \rangle - \epsilon \|\lambda\| > 0$. Then, the optimization problem can be equivalently written as

$$\begin{cases} \sup_{\lambda, \alpha > 0} & r \left(\langle \lambda, x \rangle - \epsilon \|\lambda\| \right)^q - \alpha \\ \text{subject to} & \begin{cases} \langle \lambda, \phi(h) \rangle + \delta \|\lambda\| = \alpha \\ \langle \lambda, x \rangle - \epsilon \|\lambda\| > 0. \end{cases} \end{cases}$$

Redefining new variables $\lambda' := \frac{1}{\alpha} \lambda$, the above optimization problem is written as

$$\begin{cases} \sup_{\lambda', \alpha > 0} & \alpha^q r \left(\langle \lambda', x \rangle - \epsilon \|\lambda'\| \right)^q - \alpha \\ \text{subject to} & \begin{cases} \langle \lambda', \phi(h) \rangle + \delta \|\lambda'\| = 1 \\ \langle \lambda', x \rangle - \epsilon \|\lambda'\| > 0. \end{cases} \end{cases}$$

16. $s(r, q) := \left((1 - q)(q^q r)^{\frac{1}{1-q}} \right)$

By keeping a feasible λ' fixed, one can explicitly optimize over $\alpha > 0$. In fact, for any $r' > 0$ we know that

$$\sup_{\alpha > 0} (r' \alpha^q - \alpha) = (r')^{\frac{1}{1-q}} q^{\frac{q}{1-q}} (1 - q).$$

Substituting $r' = r(\langle \lambda', x \rangle - \epsilon \|\lambda'\|)^q$, we see that (56) simplifies to

$$\begin{cases} \sup_{\lambda'} & s(r, q) \left(\langle \lambda', x \rangle - \epsilon \|\lambda'\| \right)^{\frac{q}{1-q}} \\ \text{subject to} & \begin{cases} \langle \lambda', \phi(h) \rangle + \delta \|\lambda'\| = 1 \\ \langle \lambda', x \rangle - \epsilon \|\lambda'\| > 0. \end{cases} \end{cases}$$

Since, $\eta_h < +\infty$, we know that $\langle \lambda', \phi(h) \rangle + \delta \|\lambda'\| > 0$ for every λ' satisfying $\langle \lambda', x \rangle - \epsilon \|\lambda'\| > 0$. Moreover, if $\langle \lambda', \phi(h) \rangle + \delta \|\lambda'\| < 1$ also holds for λ' , we see that its scaled version $\lambda'' := \frac{1}{\langle \lambda', \phi(h) \rangle + \delta \|\lambda'\|} \lambda'$, satisfies

$$\begin{aligned} \langle \lambda'', x \rangle - \epsilon \|\lambda''\| &> \langle \lambda', x \rangle - \epsilon \|\lambda'\| \quad \text{and} \\ \langle \lambda'', \phi(h) \rangle + \delta \|\lambda''\| &= 1. \end{aligned}$$

Therefore, the equality constraint $\langle \lambda', \phi(h) \rangle + \delta \|\lambda'\| = 1$ can be relaxed to an inequality without changing the value of the supremum. Thus, we obtain the following problem equivalent to (56).

$$\begin{cases} \sup_{\lambda'} & s(r, q) \left(\langle \lambda', x \rangle - \epsilon \|\lambda'\| \right)^{\frac{q}{1-q}} \\ \text{subject to} & \begin{cases} \langle \lambda', \phi(h) \rangle + \delta \|\lambda'\| \leq 1 \\ \langle \lambda', x \rangle - \epsilon \|\lambda'\| > 0. \end{cases} \end{cases}$$

Finally, we observe that $[0, \infty[\ni (\cdot) \mapsto (\cdot)^{\frac{q}{1-q}} \in [0, +\infty[$ is an increasing function for every $q \in]0, 1[$. Then it follows at once from Lemma 35 that the optimal value of (56) is equal to $s(r, q)(\eta_h)^{\frac{q}{1-q}}$.

Finally if $q = 1$, employing similar technique as before, reduces the problem (56) to

$$\begin{cases} \sup_{\lambda', \alpha > 0} & \alpha r \left(\langle \lambda', x \rangle - \epsilon \|\lambda'\| - \frac{1}{r} \right) \\ \text{subject to} & \begin{cases} \langle \lambda', \phi(h) \rangle + \delta \|\lambda'\| = 1 \\ \langle \lambda', x \rangle - \epsilon \|\lambda'\| > 0, \end{cases} \end{cases}$$

which simplifies to: $\sup_{\alpha > 0} \alpha r (\eta_h - \frac{1}{r})$. It then follows at once that the optimal value of the sup problem (56) is finite and equal to 0 if and only if $\eta_h \leq \frac{1}{r}$. The proof is now complete. ■

Proof [Theorem 10] Solving for the supremum over λ for every $h \in V_c$ in the min-sup problem (19), we deduce from Lemma 36 that (19) reduces to

$$\min_{h \in V_c} s(r, q) \eta_h^{\frac{q}{1-q}}.$$

Since $]0, +\infty[\ni \eta \mapsto \eta^{\frac{q}{1-q}}$ is an increasing function for every $q \in]0, 1[$, in view of the assertion (ii) of Lemma 35, we conclude that the minimization over the variable h is achieved at h^* such that $\eta_{h^*} = (C_\delta(\phi, x, \epsilon))^{1/p}$. Therefore, the optimal value of the min-sup problem (19) is equal to $s(r, q) (C_\delta(\phi, x, \epsilon))^{\frac{q}{p(1-q)}}$ and the set of minimizers is $\frac{1}{(C_\delta(\phi, x, \epsilon))^{1/p}} F_\delta(\phi, x, \epsilon)$. This establishes the assertions (i) and (ii)-(a) of the theorem.

Necessary condition for (h^*, λ^*) to be a saddle point solution.

Suppose that $(h^*, \lambda^*) \in V_c \times \mathbb{H}_n$ is a saddle point solution to the min-sup problem (19). Then necessarily, we have

$$h^* \in \operatorname{argmin}_{h \in V_c} \begin{cases} \sup_{\lambda} & r \left(\langle \lambda, x \rangle - \epsilon \|\lambda\| \right)^q - \left(\delta \|\lambda\| + \langle \lambda, \phi(h) \rangle \right) \\ \text{subject to} & \langle \lambda, x \rangle - \epsilon \|\lambda\| > 0, \end{cases}$$

which implies that $h^* \in \frac{1}{(C_\delta(\phi, x, \epsilon))^{1/p}} \cdot F_\delta(\phi, x, \epsilon)$. Moreover, we also have

$$\lambda^* \in \begin{cases} \operatorname{argmax}_{\lambda} & \min_{h \in V_c} \left\{ r \left(\langle \lambda, x \rangle - \epsilon \|\lambda\| \right)^q - \left(\delta \|\lambda\| + \langle \lambda, \phi(h) \rangle \right) \right\} \\ \text{subject to} & \langle \lambda, x \rangle - \epsilon \|\lambda\| > 0. \end{cases} \quad (57)$$

The minimization over h can be solved explicitly, and simplifying using (42), we have

$$\lambda^* \in \begin{cases} \operatorname{argmax}_{\lambda} & r \left(\langle \lambda, x \rangle - \epsilon \|\lambda\| \right)^q - \|\lambda\|'_\phi \\ \text{subject to} & \langle \lambda, x \rangle - \epsilon \|\lambda\| \geq 0. \end{cases}$$

By defining the new variables $\alpha := \|\lambda\|'_\phi$, and $\lambda' := \frac{1}{\|\lambda\|'_\phi} \lambda$, and writing the above optimization problem in terms of the variables (λ', α) , we obtain

$$\left(\frac{\lambda^*}{\|\lambda^*\|'_\phi}, \|\lambda^*\|'_\phi \right) \in \begin{cases} \operatorname{argmax}_{(\lambda', \alpha)} & r \alpha^q \left(\langle \lambda', x \rangle - \epsilon \|\lambda'\| \right)^q - \alpha \\ \text{subject to} & \begin{cases} \alpha > 0, \\ \|\lambda'\|'_\phi = 1, \\ \langle \lambda', x \rangle - \epsilon \|\lambda'\| > 0. \end{cases} \end{cases} \quad (58)$$

Observe that for every feasible λ' , the optimization over the variable α can be solved explicitly. Then from arguments similar to the ones provided in the proof of Lemma 36, we

conclude that

$$\begin{aligned}
 \frac{1}{\|\lambda^*\|'_\phi} \lambda^* &\in \begin{cases} \operatorname{argmax}_{\lambda'} & \max_{\alpha>0} \left\{ r\alpha^q \left(\langle \lambda', x \rangle - \epsilon \|\lambda'\| \right)^q - \alpha \right. \\ \text{subject to} & \left. \begin{cases} \|\lambda'\|'_\phi = 1 \\ \langle \lambda', x \rangle - \epsilon \|\lambda'\| > 0, \end{cases} \end{cases} \\
 &= \begin{cases} \operatorname{argmax}_{\lambda'} & s(r, q) \left(\langle \lambda', x \rangle - \epsilon \|\lambda'\| \right)^{\frac{q}{1-q}} \\ \text{subject to} & \begin{cases} \|\lambda'\|'_\phi = 1 \\ \langle \lambda', x \rangle - \epsilon \|\lambda'\| > 0, \end{cases} \end{cases} \\
 &= \begin{cases} \operatorname{argmax}_{\lambda'} & \langle \lambda', x \rangle - \epsilon \|\lambda'\| \\ \text{subject to} & \begin{cases} \|\lambda'\|'_\phi \leq 1 \\ \langle \lambda', x \rangle - \epsilon \|\lambda'\| > 0, \end{cases} \end{cases} \\
 &= \Lambda_\delta(\phi, x, \epsilon).^{17}
 \end{aligned}$$

Since every $\lambda \in \Lambda_\delta(\phi, x, \epsilon)$ satisfies $\langle \lambda, x \rangle - \epsilon \|\lambda\| = C_\delta(\phi, x, \epsilon)^{\frac{1}{p}}$, we conclude from (58) that the following also holds.

$$\begin{aligned}
 \|\lambda^*\|'_\phi &\in \operatorname{argmax}_{\alpha>0} \left\{ r\alpha^q C_\delta(\phi, x, \epsilon)^{\frac{q}{p}} - \alpha \right. \\
 &= (rq)^{\frac{1}{1-q}} \left(C_\delta(\phi, x, \epsilon) \right)^{\frac{q}{p(1-q)}}.
 \end{aligned}$$

Therefore, $\lambda^* \in (rq)^{\frac{1}{1-q}} \left(C_\delta(\phi, x, \epsilon) \right)^{\frac{q}{p(1-q)}} \cdot \Lambda_\delta(\phi, x, \epsilon)$, and the necessary conditions hold.

Sufficient condition for (h^*, λ^*) to be a saddle point solution.

Since $0 < (rq)^{\frac{1}{1-q}} \left(C_\delta(\phi, x, \epsilon) \right)^{\frac{q}{p(1-q)}}$, multiplying by this positive quantity throughout in (45) gives us $\langle \lambda^*, \phi(h^*) \rangle = \max_{h \in V_c} \langle \lambda^*, \phi(h) \rangle$. Then it immediately follows that

$$h^* \in \operatorname{argmin}_{h \in V_c} r \left(\langle \lambda^*, x \rangle - \epsilon \|\lambda^*\| \right)^q - \left(\delta \|\lambda^*\| + \langle \lambda^*, \phi(h) \rangle \right). \quad (59)$$

From Lemma 35(ii), we note that $\eta_{h^*} = C_\delta(\phi, x, \epsilon)^{1/p}$. Therefore, from Lemma 36 we have

$$s(r, q) C_\delta(\phi, x, \epsilon)^{\frac{q}{p(1-q)}} = \begin{cases} \sup_{\lambda} & r \left(\langle \lambda, x \rangle - \epsilon \|\lambda\| \right)^q - \left(\delta \|\lambda\| + \langle \lambda, \phi(h^*) \rangle \right) \\ \text{subject to} & \langle \lambda, x \rangle - \epsilon \|\lambda\| > 0. \end{cases}$$

Moreover, from (45) and Definition 6 of the set $\Lambda_\delta(\phi, x, \epsilon)$, it is a straightforward exercise to verify that

$$s(r, q) C_\delta(\phi, x, \epsilon)^{\frac{q}{p(1-q)}} = r \left(\langle \lambda^*, x \rangle - \epsilon \|\lambda^*\| \right)^q - \left(\delta \|\lambda^*\| + \langle \lambda^*, \phi(h^*) \rangle \right).$$

17. Since $\|x\| > \epsilon$, we know that the optimal value achieved in (17) is positive. Therefore, adding the additional constraint $\langle \lambda', x \rangle - \epsilon \|\lambda'\| > 0$ neither changes the optimizer nor the optimal value. Then the last last equality follows from Theorem 7.

Since it is obvious from the Definition 6 that $\langle \lambda^*, x \rangle - \epsilon \|\lambda^*\| > 0$, we get at once that

$$\lambda^* \in \begin{cases} \operatorname{argmax}_{\lambda} & r \left(\langle \lambda, x \rangle - \epsilon \|\lambda\| \right)^q - \left(\delta \|\lambda\| + \langle \lambda, \phi(h^*) \rangle \right) \\ \text{subject to} & \langle \lambda, x \rangle - \epsilon \|\lambda\| > 0. \end{cases} \quad (60)$$

Collecting (59) and (60), we conclude that (h^*, λ^*) is indeed a saddle point solution to (19). The proof is now complete. \blacksquare

Proof [Proposition 19] If x is not (ϕ, ϵ, δ) -feasible, then we know that $\delta = 0$ and $B[x, \epsilon] \cap \operatorname{image}(\phi) = \emptyset$. Consequently, there exists $\mathbb{H}_n \ni \lambda' \perp \operatorname{image}(\phi)$ such that $\langle \lambda', x \rangle - \epsilon \|\lambda'\| > 0$. Therefore, for every $\alpha > 0$, $\lambda_\alpha := \alpha \lambda'$ is a feasible point, and by considering arbitrarily large values of α we see that

$$+\infty = \sup_{\lambda \in \mathbb{H}_n} \left\{ c(f) + \left(\langle \lambda, x \rangle - \epsilon \|\lambda\| \right) - \langle \lambda, \phi(f) \rangle \right\}$$

for every $f \in \mathbb{R}^K$. Observe that since $\delta = 0$, the constraint $\|\lambda\| \leq \frac{1}{\delta}$ in (24) can be omitted. Thus, if x is not (ϕ, ϵ, δ) -feasible, the optimal value of the min-max problem (24) is $+\infty$.

For every $f \in \mathbb{R}^K$ define

$$\eta_f := \sup_{\|\lambda\| < \frac{1}{\delta}} c(f)(1 - \delta \|\lambda\|) + \left(\langle \lambda, x \rangle - \epsilon \|\lambda\| \right) - \langle \lambda, \phi(f) \rangle.$$

To complete the proof of assertions (1) and (ii) of the proposition, we show that for every x that is (ϕ, ϵ, δ) -feasible, we have

$$C_\delta(\phi, x, \epsilon) = \min_{f \in \mathbb{R}^K} \eta_f, \quad \text{and} \quad F_\delta(\phi, x, \epsilon) = \operatorname{argmin}_{f \in \mathbb{R}^K} \eta_f.$$

We begin by first showing that the inequality $C_\delta(\phi, x, \epsilon) \leq \eta_f$ holds for every $f \in \mathbb{R}^K$. From the Cauchy-Schwartz inequality: $\|x - \phi(f)\| = \max_{\|\lambda\| \leq 1} \langle \lambda, x - \phi(f) \rangle$, we get

$$\eta_f = c(f) + \sup_{\alpha \in [0, \frac{1}{\delta}[} \alpha \left(\|x - \phi(f)\| - (\epsilon + \delta c(f)) \right). \quad (61)$$

Case 1: If $\delta = 0$ and $\|x - \phi(f)\| > \epsilon$.

Since $\delta = 0$, α is unconstrained in the maximization problem of (61), and therefore, $\eta_f = +\infty$. Since x is (ϕ, ϵ, δ) -feasible we have $C_\delta(\phi, x, \epsilon) < +\infty$, and the inequality $C_\delta(\phi, x, \epsilon) \leq \eta_f$ follows.

Case 2: If $\delta = 0$ and $\|x - \phi(f)\| \leq \epsilon$.

It is immediate that

$$\begin{aligned} \eta_f &= c(f) + \left(\|x - \phi(f)\| - \epsilon \right) \inf_{\alpha \in [0, \frac{1}{\delta}[} \alpha \\ &= c(f). \end{aligned}$$

Recall that the LIP (8) reduces to (9), then the inequality $C_\delta(\phi, x, \epsilon) \leq \eta_f$ follows immediately from the feasibility of f in (9).

Case 3: If $\delta > 0$.

It is easily verified that $\eta_f = c(f) + \frac{1}{\delta} \max \{0, \|x - \phi(f)\| - (\epsilon + \delta c(f))\}$. Observe that $c(f) \leq \eta_f$ follows trivially, and moreover,

$$\begin{aligned} \|x - \phi(f)\| &= (\epsilon + \delta c(f)) + (\|x - \phi(f)\| - (\epsilon + \delta c(f))) \\ &\leq (\epsilon + \delta c(f)) + \max \{0, \|x - \phi(f)\| - (\epsilon + \delta c(f))\} \\ &= (\epsilon + \delta c(f)) + \delta(\eta_f - c(f)) \\ &= \epsilon + \delta \eta_f. \end{aligned}$$

Therefore, the pair (η_f, f) is a feasible point in the LIP (8), and consequently, the inequality $C_\delta(\phi, x, \epsilon) \leq \eta_f$ follows.

Let us consider $f^* \in F_\delta(\phi, x, \epsilon)$, to establish that the inequality $C_\delta(\phi, x, \epsilon) \leq \min_{f \in \mathbb{R}^K} \eta_f$ is indeed satisfied with the equality, it suffices to show that $\eta_{f^*} = C_\delta(\phi, x, \epsilon)$. If $\delta = 0$, then indeed $\eta_{f^*} = c(f^*) = C_\delta(\phi, x, \epsilon)$. If $\delta > 0$, we know that $\eta_{f^*} \geq c(f^*)$, whereby we have

$$\|x - \phi(f^*)\| - (\epsilon + \delta c(f^*)) \geq \|x - \phi(f^*)\| - (\epsilon + \delta \eta_{f^*}) \geq 0.$$

Using $\eta_{f^*} = c(f^*) + \frac{1}{\delta} \max \{0, \|x - \phi(f^*)\| - (\epsilon + \delta c(f^*))\}$ and simplifying, we get

$$\eta_{f^*} = \frac{1}{\delta} (\|x - \phi(f^*)\| - \epsilon) = C_\delta(\phi, x, \epsilon), \quad (62)$$

where the last equality follows from the assertion (i) of Lemma 25. Furthermore, if there exists $f' \in \mathbb{R}^K$ such that $\eta_{f'} = C_\delta(\phi, x, \epsilon)$, we know that the pair $(\eta_{f'}, f')$ is a feasible point in the LIP (8) it readily follows that $f' \in F_\delta(\phi, x, \epsilon)$. This completes the proof of assertions (i) and (ii) of the proposition.

Necessary condition for the pair (f^*, λ^*) to be a saddle point in (24).

The fact that $f^* \in F_\delta(\phi, x, \epsilon)$ follows at once from assertion (ii) of the proposition. To prove that $\lambda^* \in \Lambda_\delta(\phi, x, \epsilon)$, first we observe that

$$\begin{aligned} \min_{f \in \mathbb{R}^K} c(f)(1 - \delta \|\lambda\|) - \langle \lambda, \phi(f) \rangle &= \min_{\eta \geq 0, h \in V_c} \eta(1 - \delta \|\lambda\|) - \langle \lambda, \phi(\eta h) \rangle \\ &= \min_{\eta \geq 0} \eta(1 - \|\lambda\|'_\phi) \quad (\text{from (42)}) \\ &= \begin{cases} -\infty & \text{if } \|\lambda\|'_\phi > 1, \\ 0 & \text{if } \|\lambda\|'_\phi \leq 1. \end{cases} \end{aligned} \quad (63)$$

Therefore, if (f^*, λ^*) is a saddle point in (24), we have

$$\begin{aligned}
\lambda^* &\in \operatorname{argmax}_{\|\lambda\| < \frac{1}{\delta}} \left\{ \min_{f \in \mathbb{R}^K} c(f)(1 - \delta \|\lambda\|) + (\langle \lambda, x \rangle - \epsilon \|\lambda\|) - \langle \lambda, \phi(f) \rangle \right. \\
&\in \operatorname{argmax}_{\substack{\|\lambda\| < \frac{1}{\delta}, \\ \|\lambda\|'_\phi \leq 1}} \langle \lambda, x \rangle - \epsilon \|\lambda\| \quad (\text{from (63)}) \\
&\in \operatorname{argmax}_{\|\lambda\|'_\phi \leq 1} \langle \lambda, x \rangle - \epsilon \|\lambda\| \quad (\text{because, } \delta \|\lambda\| \leq \|\lambda\|'_\phi, \text{ from (42)}) \\
&\in \Lambda_\delta(\phi, x, \epsilon) \quad (\text{from Theorem 7 (iii, b)}).
\end{aligned}$$

Sufficient condition for the pair (f^*, λ^*) to be a saddle point in (24).

Let $\lambda^* \in \Lambda_\delta(\phi, x, \epsilon)$ and $f^* \in F_\delta(\phi, x, \epsilon)$. From Definition 6, we see that $\|\lambda^*\|'_\phi = 1$ and $(1 - \delta \|\lambda^*\|) \geq 0$. Therefore, from (63)

$$\begin{aligned}
0 &= \min_{f \in \mathbb{R}^K} c(f)(1 - \delta \|\lambda^*\|) - \langle \lambda^*, \phi(f) \rangle \\
&\leq c(f^*)(1 - \delta \|\lambda^*\|) - \langle \lambda^*, \phi(f^*) \rangle \\
&\leq C_\delta(\phi, x, \epsilon)(1 - \delta \|\lambda^*\|) - \langle \lambda^*, \phi(h^*) \rangle \\
&= 0,
\end{aligned}$$

where $h^* := \frac{1}{C_\delta(\phi, x, \epsilon)} f^*$, and the last equality follows from (45). Therefore, all the inequalities are satisfied with equality, and we have $0 = c(f^*)(1 - \delta \|\lambda^*\|) - \langle \lambda^*, \phi(f^*) \rangle$. Moreover, it also immediately implies that

$$f^* \in \operatorname{argmin}_{f \in \mathbb{R}^K} c(f)(1 - \delta \|\lambda^*\|) + (\langle \lambda^*, x \rangle - \epsilon \|\lambda^*\|) - \langle \lambda^*, \phi(f) \rangle. \quad (64)$$

From Definition 6 we know that $\langle \lambda^*, x \rangle - \epsilon \|\lambda^*\| = C_\delta(\phi, x, \epsilon)$ and from (42) we have $\delta \|\lambda^*\| = 1 - \max_{h \in V_c} \langle \lambda^*, \phi(h) \rangle \leq 1$. Moreover, from the fact that $0 = c(f^*)(1 - \delta \|\lambda^*\|) - \langle \lambda^*, \phi(f^*) \rangle$ we have

$$\left(c(f^*)(1 - \delta \|\lambda^*\|) - \langle \lambda^*, \phi(f^*) \rangle \right) + \left(\langle \lambda^*, x \rangle - \epsilon \|\lambda^*\| \right) = 0 + C_\delta(\phi, x, \epsilon).$$

Recalling from (62) that $\eta_{f^*} = C_\delta(\phi, x, \epsilon)$, we immediately get

$$\lambda^* \in \operatorname{argmax}_{\|\lambda\| \leq \frac{1}{\delta}} c(f^*)(1 - \delta \|\lambda\|) + (\langle \lambda, x \rangle - \epsilon \|\lambda\|) - \langle \lambda, \phi(f^*) \rangle. \quad (65)$$

Collecting (64) and (65), we conclude that $(f^*, \lambda^*) \in F_\delta(\phi, x, \epsilon) \times \Lambda_\delta(\phi, x, \epsilon)$ is indeed a saddle point solution to the min-max problem (24), and the proof is now complete. \blacksquare

4. Conclusion

In this article, we have proposed a slightly generalised formulation of the error constrained linear inverse problem and provide an exposition to its underlying convex geometry. Novel convex-concave min-max problems have been proposed and their equivalence to the LIP is mathematically established. These equivalent reformulations are crucial in overcoming the ill-posedness of the error constrained dictionary learning problem. Furthermore, complete characterization of the saddle points of the min-max problems is also provided in terms of a solution to the LIP, and vice versa. Consequently, a solution to the LIP can be computed by applying saddle point seeking methods to its equivalent min-max problems, which gives rise to simple algorithms to solve linear inverse problems and problems alike. Of course, the intent of this article is to only show that the min-max forms can also be used to obtain algorithms for an LIP, comparison of the resulting algorithms with the existing methods needs a separate and thorough investigation of its own, and will be reported in subsequent articles.

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