Fundamental Conditions for Low-CP-Rank Tensor Completion

Morteza Ashraphijuo   ASHRAPHIJUO@EE.COLUMBIA.EDU
Columbia University
New York, NY 10027, USA

Xiaodong Wang   WANGX@EE.COLUMBIA.EDU
Columbia University
New York, NY 10027, USA

Abstract

We consider the problem of low canonical polyadic (CP) rank tensor completion. A completion is a tensor whose entries agree with the observed entries and its rank matches the given CP rank. We analyze the manifold structure corresponding to the tensors with the given rank and define a set of polynomials based on the sampling pattern and CP decomposition. Then, we show that finite completability of the sampled tensor is equivalent to having a certain number of algebraically independent polynomials among the defined polynomials. Our proposed approach results in characterizing the maximum number of algebraically independent polynomials in terms of a simple geometric structure of the sampling pattern, and therefore we obtain the deterministic necessary and sufficient condition on the sampling pattern for finite completability of the sampled tensor. Moreover, assuming that the entries of the tensor are sampled independently with probability p and using the mentioned deterministic analysis, we propose a combinatorial method to derive a lower bound on the sampling probability p, or equivalently, the number of sampled entries that guarantees finite completability with high probability. We also show that the existing result for the matrix completion problem can be used to obtain a loose lower bound on the sampling probability p. In addition, we obtain deterministic and probabilistic conditions for unique completability. It is seen that the number of samples required for finite or unique completability obtained by the proposed analysis on the CP manifold is orders-of-magnitude lower than that is obtained by the existing analysis on the Grassmannian manifold.

Keywords: Low-rank tensor completion, canonical polyadic (CP) decomposition, finite completability, unique completability, algebraic geometry, Bernstein's theorem.

1. Introduction

The fast progress in data science and technology has given rise to the extensive applications of multi-way datasets, which allow us to take advantage of the inherent correlations across different attributes. The classical matrix analysis limits its efficiency in exploiting the correlations across different features from a multi-way perspective. In contrast, analysis of multi-way data (tensor), which was originally proposed in the field of psychometrics and recently found applications in machine learning and signal processing, is capable of taking full advantage of these correlations (De Lathauwer, 2009; Kolda and Bader, 2009; Abraham et al., 2012; Grasedyck et al., 2013; Muti and Bourennane, 2007; Signoretto et al., 2011). The problem of low-rank tensor completion, i.e., reconstructing a tensor from a subset of its entries given the rank, which is generally NP hard (Hillar and Lim, 2013), arises in compressed sensing (Lim and Comon, 2010; Sidiropoulos and

Kyrillidis, 2012; Gandy et al., 2011), visual data reconstruction (Liu et al., 2013, 2016), seismic data processing (Kreimer et al., 2013; Ely et al., 2013), etc. Existing approaches to low-rank data completion mainly focus on convex relaxation of matrix rank (Candès and Recht, 2009; Candès and Tao, 2010; Cai et al., 2010; Ashraphijuo et al., 2016b; Candès et al., 2013; Ashraphijuo et al., 2015) or different convex relaxations of tensor ranks (Gandy et al., 2011; Tomioka et al., 2010; Signoretto et al., 2014; Romera-Paredes and Pontil, 2013). In addition, there are some other works which provide a completion of the given rank using alternating minimization, e.g., for low-CP-rank tensor (Jain and Oh, 2014) and low-TT-rank tensor (Wang et al., 2016).

Tensors consisting of real-world datasets usually have a low rank structure. The manifold of low-rank tensors has recently been investigated in several works (Grasedyck et al., 2013; Ashraphijuo et al., 2016a; Ashraphijuo and Wang, 2017). In this paper, we focus on the canonical polyadic (CP) decomposition (Harshman, 1970; ten Berge and Sidiropoulos, 2002; Carroll and Chang, 1970; Stegeman and Sidiropoulos, 2007) and the corresponding CP rank, but in general there are other well-known tensor decompositions including Tucker decomposition (Kolda, 2001; Grasedyck, 2010; Kressner et al., 2014), tensor-train (TT) decomposition (Oseledets, 2011; Holtz et al., 2012), tubal rank decomposition (Kilmer et al., 2013) and several other methods (De Lathauwer et al., 2000; Papalexakis et al., 2012). Note that most existing literature on tensor completion based on various optimization formulations use CP rank (Gandy et al., 2011; Krishnamurthy and Singh, 2013).

Deterministic conditions on the locations of the sampled entries (sampling pattern) are obtained through algebraic geometry analyses on Grassmannian manifold that lead to finite/unique solutions to the matrix completion problem (Pimentel-Alarcón et al., 2016d). Also, deterministic conditions on the sampling patterns have been studied for subspace clustering in (Pimentel-Alarcón et al., 2016c, 2015, 2016a,b). In this paper, we study the fundamental conditions on the sampling pattern to ensure finite or unique number of completions, where these fundamental conditions are independent of the correlations of the entries of the tensor, in contrast to the common assumption adopted in literature such as incoherence. Given the rank of a matrix, Pimentel-Alarcón *et. al.* in (Pimentel-Alarcón et al., 2016d) obtains such fundamental conditions on the sampling pattern for finite completability of the matrix. Previously, we treated the same problem for multi-view matrix (Ashraphijuo et al., 2017c,b), tensor given its Tucker rank (Ashraphijuo et al., 2016a, 2017a), and tensor given its TT rank(Ashraphijuo and Wang, 2017). In this paper, the structure of the CP decomposition and the geometry of the corresponding manifold are investigated to obtain the fundamental conditions for finite completability given its CP rank. These recently developed algebraic geometry analyses can also be used to approximate the rank of partially sampled data (Ashraphijuo et al., 2017d).

To emphasize the contribution of this work, we highlight the differences and challenges in comparison with the Tucker and TT tensor models. In CP decomposition, the notion of tensor multiplication is different from those for Tucker and TT, and therefore the geometry of the manifold and the algebraic variety are completely different. Moreover, in CP decomposition we are dealing with the sum of several tensor products, which is not the case in Tucker and TT decompositions, and therefore the equivalence classes or geometric patterns that are needed to study the algebraic variety are different. Moreover, CP rank is a scalar and the ranks of matricizations and unfoldings are not given in contrast with the Tucker and TT models.

Let $\mathcal{U}$ denote the sampled tensor and Ω denote the binary sampling pattern tensor that is of the same dimension and size as $\mathcal{U}$. The entries of Ω that correspond to the observed entries of $\mathcal{U}$ are equal to 1 and the rest of the entries are set as 0. Assume that the entries of $\mathcal{U}$ are sampled

2

independently with probability p. This paper is mainly concerned with treating the following three problems.

Problem (i): Given the CP rank, characterize the necessary and sufficient conditions on the sampling pattern Ω, under which there exist only finitely many completions of $\mathcal{U}$.

We consider the CP decomposition of the sampled tensor, where all rank-1 tensors in this decomposition are unknown and we only have some entries of $\mathcal{U}$. Then, each sampled entry results in a polynomial such that the variables of the polynomial are the entries of the rank-1 tensors in the CP decomposition. We propose a novel analysis on the CP manifold to obtain the maximum number of algebraically independent polynomials, among all polynomials corresponding to the sampled entries, in terms of the geometric structure of the sampling pattern Ω. We show that if the maximum number of algebraically independent polynomials is a given number, then the sampled tensor $\mathcal{U}$ is finitely completable. Due to the fundamental differences between the CP decomposition and the Tucker or TT decomposition, this analysis is completely different from our previous works (Ashraphijuo et al., 2016a; Ashraphijuo and Wang, 2017). Moreover, note that our proposed algebraic geometry analysis on the CP manifold is not a simple generalization of the existing analysis on the Grassmannian manifold (Pimentel-Alarcón et al., 2016d) even though the CP decomposition is a generalization of rank factorization of a matrix, as almost every step needs to be developed anew.

Problem (ii): Characterize conditions on the sampling pattern to ensure that there is exactly one completion for the given CP rank.

Similar to Problem (i), our approach is to study the algebraic independence of the polynomials corresponding to the samples. We exploit the properties of a set of minimally algebraically dependent polynomials to add additional constraints on the sampling pattern such that each of the rank-1 tensors in the CP decomposition can be determined uniquely.

As we will observe later, the deterministic conditions for finite or unique completability are combinatorial in nature, which are hard to verify if they hold true in practice. Therefore, we also provide a probabilistic analysis to verify the validity of such conditions based on the number of samples or the sampling probability. However, we cannot guarantee the validity of the constraints with probability one anymore (i.e., not deterministically anymore) and instead we show that the combinatorial conditions hold true with high probability if the sampling probability is above certain threshold.

Problem (iii): Provide a lower bound on the total number of sampled entries or the sampling probability p such that the proposed conditions on the sampling pattern Ω for finite and unique completability are satisfied with high probability.

We develop several combinatorial tools together with our previous graph theory results in (Ashraphijuo et al., 2016a) to obtain lower bounds on the total number of sampled entries, i.e., lower bounds on the sampling probability p, such that the deterministic conditions for Problems (i) and (ii) are met with high probability. Particularly, it is shown in (Krishnamurthy and Singh, 2013), $\mathcal{O}(nr^{\frac{d-1}{2}}d^2\log(r))$ samples are required to recover the tensor $\mathcal{U} \in \mathbb{R}^{\overbrace{n \times \cdots \times n}^{d}}$ of rank r. Recall that in this paper, we obtain the number samples to ensure finite/unique completability that is independent of the completion algorithm. As we show later, using the existing analysis on the Grassmannian manifold results in $\mathcal{O}(n^{\frac{d+1}{2}}\max\{d\log(n) + \log(r), r\})$ samples to ensure finite/unique completability. However, our proposed analysis on the CP manifold results in $\mathcal{O}(n^2\max\{\log(nrd), r\})$ samples to guarantee the finiteness of the number of completions, which is significantly lower than that given in (Krish-

3

namurthy and Singh, 2013). Hence, the fundamental conditions for tensor completion motivate new optimization formulation to close the gap in the number of required samples.

Tucker decomposition consists of a core tensor multiplied by a matrix along each dimension. TT decomposition of a d-way tensor consists of the train-wise multiplication of a matrix and $d-2$ three-way tensors and finally another matrix. Tucker and TT ranks are defined as the vectors consisting of rank of matricizations and rank of unfoldings, respectively. The structure of the manifold of the tensors with a fixed Tucker (or TT) rank vector is fundamentally different with that of the CP manifold.

Note that the general framework of using rank factorization to study the algebraic independence of the polynomials through counting the number of involved variables in the polynomials is indeed similar to that in (Pimentel-Alarcón et al., 2016d; Ashraphijuo et al., 2016a; Ashraphijuo and Wang, 2017) to some extent. However, since the manifold considered in this paper (CP manifold) is fundamentally different from the Grassmannian or Tucker or TT manifold, all results in this paper are original. In particular, we mention some of the main differences: (i) geometry of the manifold, (ii) the equivalence class for the basis and consequently (iii) the canonical basis, (iv) the structure of the polynomials, etc., are fundamentally different from those in the literature. Hence, the deterministic analyses are fundamentally different (excluding Lemma 7). As a consequence, the probabilistic analysis is also basically different as the combinatorial condition (6) is obtained based on the defined geometry and equivalence class for CP decomposition. However, some lemmas including the generalization of Hall's theorem for bipartite graphs (Lemma 18) or Lemma 17 that applies pigeonhole principle are taken from our previous papers (Ashraphijuo et al., 2016a; Ashraphijuo and Wang, 2017) in probabilistic analysis.

The remainder of this paper is organized as follows. In Section 2, the preliminaries and problem statement are presented. In Section 3, we develop necessary and sufficient deterministic conditions for finite completability. In Section 4, we develop probabilistic conditions for finite completability. In Section 5, we consider unique completability and obtain both deterministic and probabilistic conditions. Some numerical results are provided in Section 6. Finally, Section 7 concludes the paper.

2. Background

2.1 Preliminaries and Notations

In this paper, it is assumed that a d-way tensor $\mathcal{U} \in \mathbb{R}^{n_1 \times \cdots \times n_d}$ is sampled. Throughout this paper, we use CP rank as the rank of a tensor, which is defined as the minimum number r such that there exist $\mathbf{a}_i^l \in \mathbb{R}^{n_i}$ for $1 \leq i \leq d$ and $1 \leq l \leq r$ and

$$\mathcal{U} = \sum_{l=1}^{r} \mathbf{a}_1^l \otimes \mathbf{a}_2^l \otimes \ldots \otimes \mathbf{a}_d^l, \tag{1}$$

or equivalently,

$$\mathcal{U}(x_1, x_2, \ldots, x_d) = \sum_{l=1}^{r} \mathbf{a}_1^l(x_1)\mathbf{a}_2^l(x_2)\ldots\mathbf{a}_d^l(x_d), \tag{2}$$

where $\otimes$ denotes the tensor product (outer product) and $\mathcal{U}(x_1, x_2, \ldots, x_d)$ denotes the entry of the sampled tensor with coordinates $\vec{x} = (x_1, x_2, \ldots, x_d)$ and $\mathbf{a}_i^l(x_i)$ denotes the x_i-th entry of vector $\mathbf{a}_i^l$. Note that $\mathbf{a}_1^l \otimes \mathbf{a}_2^l \otimes \ldots \otimes \mathbf{a}_d^l \in \mathbb{R}^{n_1 \times \cdots \times n_d}$ is a rank-1 tensor, $l = 1, 2, \ldots, r$.

Denote Ω as the binary sampling pattern tensor that is of the same size as $\mathcal{U}$ and $\Omega(\vec{x}) = 1$ if $\mathcal{U}(\vec{x})$ is observed and $\Omega(\vec{x}) = 0$ otherwise. Also define $x^+ \triangleq \max\{0, x\}$.

For a nonempty $I \subset \{1, \ldots, d\}$, define $N_I \triangleq \Pi_{i \in I} \, n_i$ and also denote $\bar{I} \triangleq \{1, \ldots, d\} \backslash I$. Let $\widetilde{\mathbf{U}}_{(I)} \in \mathbb{R}^{N_I \times N_{\bar{I}}}$ be the unfolding of the tensor $\mathcal{U}$ corresponding to the set I such that $\mathcal{U}(\vec{x}) = \widetilde{\mathbf{U}}_{(I)}(\widetilde{M_I}(x_{i_1}, \ldots, x_{i_{|I|}}), \widetilde{M_{\bar{I}}}(x_{i_{|I|+1}}, \ldots, x_{i_d}))$, where $I = \{i_1, \ldots, i_{|I|}\}$, $\bar{I} = \{i_{|I|+1}, \ldots, i_d\}$, $\widetilde{M_I} : (x_{i_1}, \ldots, x_{i_{|I|}}) \to \{1, 2, \ldots, N_I\}$ and $\widetilde{M_{\bar{I}}} : (x_{i_{|I|+1}}, \ldots, x_{i_d}) \to \{1, 2, \ldots, \bar{N}_{\bar{I}}\}$ are two bijective mappings. For $i \in \{1, \ldots, d\}$ and $I = \{i\}$, we denote the unfolding corresponding to I by $\mathbf{U}_{(i)}$ and we call it the i-th matricization of tensor $\mathcal{U}$.

2.2 Problem Statement and A Motivating Example

We are interested in finding necessary and sufficient deterministic conditions on the sampling pattern tensor Ω under which there are infinite, finite, or unique completions of the sampled tensor $\mathcal{U}$ that satisfy the given CP rank constraint. Furthermore, we are interested in finding probabilistic conditions on the number of samples or the sampling probability that ensure the obtained deterministic conditions for finite and unique completability hold with high probability.

To motivate our proposed analysis in this paper on the CP manifold, we compare the following two approaches using a simple example to emphasize the exigency of our proposed analysis: (i) analyzing each of the unfoldings individually, (ii) analyzing based on the CP decomposition.

Consider a three-way tensor $\mathcal{U} \in \mathbb{R}^{2 \times 2 \times 2}$ with CP rank of 1. Assume that the entries $(1, 1, 1)$, $(2, 1, 1)$, $(1, 2, 1)$ and $(1, 1, 2)$ are observed. As a result of Lemma 11 in this paper, all unfoldings of this tensor are rank-1 matrices. It is shown in Section II of (Ashraphijuo et al., 2016a) that having any 4 entries of a rank-1 2×4 matrix, there are infinitely many completions for it. As a result, any unfolding of $\mathcal{U}$ is infinitely many completable given only the corresponding rank constraint. Next, using the CP decomposition (1), we show that there are only finitely many completions of the sampled tensor of CP rank 1.

Define $\mathbf{a}_1^1 = [x \; x']^\top \in \mathbb{R}^2$, $\mathbf{a}_2^1 = [y \; y']^\top \in \mathbb{R}^2$ and $\mathbf{a}_3^1 = [z \; z']^\top \in \mathbb{R}^2$. Then, according to (1), we have the following

$$
\begin{aligned}
\mathcal{U}(1, 1, 1) &= xyz, & \mathcal{U}(2, 2, 1) &= x'y'z, & (3) \\
\mathcal{U}(2, 1, 1) &= x'yz, & \mathcal{U}(2, 1, 2) &= x'yz', \\
\mathcal{U}(1, 2, 1) &= xy'z, & \mathcal{U}(1, 2, 2) &= xy'z', \\
\mathcal{U}(1, 1, 2) &= xyz', & \mathcal{U}(2, 2, 2) &= x'y'z'.
\end{aligned}
$$

Recall that $(1,1,1), (2,1,1), (1,2,1)$, and $(1,1,2)$ are the observed entries. Hence, the unknown entries can be determined uniquely in terms of the 4 observed entries as

$$
\begin{aligned}
\mathcal{U}(2,2,1) &= x'y'z = \frac{\mathcal{U}(2,1,1)\mathcal{U}(1,2,1)}{\mathcal{U}(1,1,1)}, \\
\mathcal{U}(2,1,2) &= x'yz' = \frac{\mathcal{U}(2,1,1)\mathcal{U}(1,1,2)}{\mathcal{U}(1,1,1)}, \\
\mathcal{U}(1,2,2) &= xy'z' = \frac{\mathcal{U}(1,2,1)\mathcal{U}(1,1,2)}{\mathcal{U}(1,1,1)}, \\
\mathcal{U}(2,2,2) &= x'y'z' = \frac{\mathcal{U}(2,1,1)\mathcal{U}(1,2,1)\mathcal{U}(1,1,2)}{\mathcal{U}(1,1,1)\mathcal{U}(1,1,1)}.
\end{aligned}
\tag{4}
$$

Therefore, based on the CP decomposition, the sampled tensor $\mathcal{U}$ is finitely (uniquely) many completable. Hence, this example illustrates that collapsing a tensor into a matrix results in loss of information and thus motivate the investigation of the tensor CP manifold.

3. Deterministic Conditions for Finite Completability

In this section, we characterize the necessary and sufficient condition on the sampling pattern for finite completability of the sampled tensor given its CP rank. In Section 3.1, we define a polynomial based on each observed entry and through studying the geometry of the manifold of the corresponding CP rank, we transform the problem of finite completability of $\mathcal{U}$ to the problem of including enough number of algebraically independent polynomials among the defined polynomials for the observed entries. In Section 3.2, a binary tensor is constructed based on the sampling pattern Ω, which allows us to study the algebraic independence of a subset of polynomials among all defined polynomials based on the samples. In Section 3.3, we characterize the connection between the maximum number of algebraically independent polynomials among all the defined polynomials and finite completability of the sampled tensor.

3.1 Geometry

Suppose that the sampled tensor $\mathcal{U}$ is chosen generically from the manifold of the tensors in $\mathbb{R}^{n_1 \times \cdots \times n_d}$ of rank r. Assume that $\mathbf{a}_i^l$ vectors are unknown for $1 \leq i \leq d$ and $1 \leq l \leq r$. For notational simplicity, define the tuples $\mathbb{A}_l = (\mathbf{a}_1^l, \mathbf{a}_2^l, \ldots, \mathbf{a}_d^l)$ for $l = 1, \ldots, r$ and $\mathbb{A} = (\mathbb{A}_1, \ldots, \mathbb{A}_r)$. Moreover, define $\mathbf{A}_i = [\mathbf{a}_i^1 | \mathbf{a}_i^2 | \ldots | \mathbf{a}_i^r] \in \mathbb{R}^{n_i \times r}$. Note that each of the sampled entries results in a polynomials in terms of the entries of $\mathbb{A}$ as in (2).

Here, we briefly mention the following two facts to highlight the fundamentals of our proposed analysis.

- **Fact** 1: As it can be seen from (2), any observed entry $\mathcal{U}(\vec{x})$ results in an equation that involves one entry of $\mathbf{a}_i^l$, $i = 1, \ldots, d$ and $l = 1, \ldots, r$. Considering the entries of $\mathbb{A}$ as variables (right-hand side of (2)), each observed entry results in a polynomial in terms of these variables. Moreover, for any observed entry $\mathcal{U}(\vec{x})$, the value of x_i specifies the location of the entry of $\mathbf{a}_i^l$ that is involved in the corresponding polynomial, $i = 1, \ldots, d$ and $l = 1, \ldots, r$.

- **Fact** 2: It can be concluded from Bernstein's theorem (Sturmfels, 2002) that in a system of n polynomials in n variables with coefficients chosen generically, the polynomials are

algebraically independent with probability one, and therefore there exist only finitely many solutions.

The following assumption will be used frequently in this paper.

Assumption 1: Each row of the d-th matricization of the sampled tensor, i.e., $\mathbf{U}_{(d)}$ includes at least r observed entries.

Observe that each row of the d-th matricization of the sampled tensor includes $n_1 \times n_2 \times \ldots \times n_{d-1}$ entries. In order to show that Assumption 1 is a very mild assumption, as an example assume that $n_1 = n_2 = \cdots = n_{d-1} = n$. Then, each row of $\mathbf{U}_{(d)}$ has n^{d-1} entries and Assumption 1 requires r sampled entries among them. As we consider the low-rank scenario, we show in Section 4 how realistic this assumption is in terms of the sampling probability.

Lemma 1 *Given $\mathbf{A}_i$'s for $i = 1, \ldots, d - 1$ and Assumption 1, $\mathbf{A}_d$ can be determined uniquely.*

Proof Each row of $\mathbf{A}_d$ has r entries and also as it can be seen from (2), each observed entry in the i-th row of $\mathbf{U}_{(d)}$ results in a degree-1 polynomial in terms of the r entries of the i-th row of $\mathbf{A}_d$. Since Assumption 1 holds, for each row of $\mathbf{A}_d$ that has r variables, we have at least r degree-1 polynomials. Genericity of the coefficients of these polynomials results that each row of $\mathbf{A}_d$ can be determined uniquely. ∎

As a result of Lemma 1, we can obtain $\mathbf{A}_d$ in terms of the entries of $\mathbf{A}_i$'s for $i = 1, \ldots, d-1$. As mentioned earlier, each observed entry is equivalent to a polynomial in the format of (2). Consider all such polynomials excluding those that have been used to obtain $\mathbf{A}_d$ (r samples at each row of $\mathbf{U}_{(d)}$) and denote this set of polynomials in terms of the entries of $\mathbf{A}_i$'s for $i = 1, \ldots, d - 1$ by $\mathcal{P}(\Omega)$.

We are interested in defining an equivalence class such that each class includes exactly one of the decompositions among all rank-r decompositions of a particular tensor and the pattern in Lemma 3 characterizes such an equivalence class. Lemma 2 below is a re-statement of Lemma 1 in (Ashraphijuo and Wang, 2017), which characterizes such an equivalence class or equivalently geometric pattern for a matrix instead of tensor. This lemma will be used to show Lemma 3 later.

Lemma 2 *Let $\mathbf{X}$ denote a generically chosen matrix from the manifold of $n_1 \times n_2$ matrices of rank r and also $\mathbf{Q} \in \mathbb{R}^{r \times r}$ be a given full rank matrix. Then, there exists a unique decomposition $\mathbf{X} = \mathbf{YZ}$ such that $\mathbf{Y} \in \mathbb{R}^{n_1 \times r}$, $\mathbf{Z} \in \mathbb{R}^{r \times n_2}$ and $\mathbf{P} = \mathbf{Q}$, where $\mathbf{P} \in \mathbb{R}^{r \times r}$ represents a submatrix[1] of $\mathbf{Y}$.*

In Lemma 3, we generalize Lemma 2 and characterize the similar pattern for a multi-way tensor. Assuming that $\mathbf{P}$ represents the submatrix of $\mathbf{Y}$ consists of the first r columns and the first r rows of $\mathbf{Y}$ and also $\mathbf{Q}$ is equal to the $r \times r$ identity matrix, this pattern is called the canonical decomposition of $\mathbf{X}$. The canonical decomposition is shown for a rank-2 matrix as the following

$$
\begin{array}{|c|c|c|c|}
\hline
1 & -1 & 0 & -1 \\
\hline
2 & 2 & 4 & 6 \\
\hline
-1 & 3 & 2 & 5 \\
\hline
1 & 2 & 3 & 5 \\
\hline
\end{array}
=
\begin{array}{|c|c|}
\hline
1 & 0 \\
\hline
0 & 1 \\
\hline
y_1 & y_2 \\
\hline
y_3 & y_4 \\
\hline
\end{array}
\times
\begin{array}{|c|c|c|c|}
\hline
x_1 & x_2 & x_3 & x_4 \\
\hline
x_5 & x_6 & x_7 & x_8 \\
\hline
\end{array}
,
$$

---

1. Specified by a subset of rows and a subset of columns (not necessarily consecutive).

where x_i's and y_i's can be determined uniquely as

$$
\begin{array}{|c|c|}
\hline
y_1 & y_2 \\
\hline
y_3 & y_4 \\
\hline
\end{array}
=
\begin{array}{|c|c|}
\hline
-2 & \frac{1}{2} \\
\hline
-\frac{1}{2} & \frac{3}{4} \\
\hline
\end{array}
\text{ and }
\begin{array}{|c|c|c|c|}
\hline
x_1 & x_2 & x_3 & x_4 \\
\hline
x_5 & x_6 & x_7 & x_8 \\
\hline
\end{array}
=
\begin{array}{|c|c|c|c|}
\hline
1 & -1 & 0 & -1 \\
\hline
2 & 2 & 4 & 6 \\
\hline
\end{array}.
$$

Also, the above canonical decomposition can be written as the following

$$
\begin{array}{|c|c|c|c|}
\hline
1 & -1 & 0 & -1 \\
\hline
2 & 2 & 4 & 6 \\
\hline
-1 & 3 & 2 & 5 \\
\hline
1 & 2 & 3 & 5 \\
\hline
\end{array}
=
\begin{array}{|c|}
\hline
1 \\
\hline
0 \\
\hline
y_1 \\
\hline
y_3 \\
\hline
\end{array}
\times
\begin{array}{|c|c|c|c|}
\hline
x_1 & x_2 & x_3 & x_4 \\
\hline
\end{array}
+
\begin{array}{|c|}
\hline
0 \\
\hline
1 \\
\hline
y_2 \\
\hline
y_4 \\
\hline
\end{array}
\times
\begin{array}{|c|c|c|c|}
\hline
x_5 & x_6 & x_7 & x_8 \\
\hline
\end{array}.
$$

Generalization of the canonical decomposition for multi-way tensor is as the following

$$
\mathbf{a}_1^1 =
\begin{array}{|c|}
\hline
1 \\
\hline
0 \\
\hline
\vdots \\
\hline
0 \\
\hline
\mathbf{a}_1^1(r+1) \\
\hline
\vdots \\
\hline
\mathbf{a}_1^1(n_1) \\
\hline
\end{array}
, \ldots , \ \mathbf{a}_1^r =
\begin{array}{|c|}
\hline
0 \\
\hline
0 \\
\hline
\vdots \\
\hline
1 \\
\hline
\mathbf{a}_1^r(r+1) \\
\hline
\vdots \\
\hline
\mathbf{a}_1^r(n_1) \\
\hline
\end{array}
,
$$

and for $i \in \{2, \ldots, d-1\}$

$$
\mathbf{a}_i^1 =
\begin{array}{|c|}
\hline
1 \\
\hline
\mathbf{a}_i^1(2) \\
\hline
\vdots \\
\hline
\mathbf{a}_i^1(n_i) \\
\hline
\end{array}
, \ldots , \ \mathbf{a}_i^r =
\begin{array}{|c|}
\hline
1 \\
\hline
\mathbf{a}_i^r(2) \\
\hline
\vdots \\
\hline
\mathbf{a}_i^r(n_i) \\
\hline
\end{array}.
$$

Lemma 3 *Let $j \in \{1, \ldots, d-1\}$ be a fixed number and define $J = \{1, \ldots, d-1\}\setminus\{j\}$. Assume that the full rank matrix $\mathbf{Q}_j \in \mathbb{R}^{r \times r}$ and matrices $\mathbf{Q}_i \in \mathbb{R}^{1 \times r}$ with nonzero entries for $i \in J$ are given. Also, let $\mathbf{P}_i$ denote an arbitrary submatrix of $\mathbf{A}_i$, $i = 1, 2, \ldots, d-1$, where $\mathbf{P}_j \in \mathbb{R}^{r \times r}$ and $\mathbf{P}_i \in \mathbb{R}^{1 \times r}$ for $i \in J$. Then, with probability one, there exists exactly one rank-r decomposition of $\mathcal{U}$ such that $\mathbf{P}_i = \mathbf{Q}_i$, $i = 1, \ldots, d-1$.*

Proof First we claim that there exists at most one rank-r decomposition of $\mathcal{U}$ such that $\mathbf{P}_i = \mathbf{Q}_i$, $i = 1, \ldots, d-1$. We assume that $\mathbf{P}_i = \mathbf{Q}_i$, $i = 1, \ldots, d-1$ and also $\mathcal{U}$ is given. Then, it suffices to show that the rest of the entries of $\mathbb{A}$ can be determined in at most a unique way (no more than one solution) in terms of the given parameters such that (1) holds. Note that if a variable can be determined uniquely through two different ways (two sets of samples or equations), in general either it can be determined uniquely if both ways result in the same value or it does not have any solution otherwise. Let y_i denote the row number of submatrix $\mathbf{P}_i \in \mathbb{R}^{1 \times r}$ for $i \in J$ and $Y_j = \{y_j^1, \ldots, y_j^r\}$ denote the row numbers of submatrix $\mathbf{P}_j \in \mathbb{R}^{r \times r}$.

As the first step of proving our claim, we show that $\mathbf{A}_d$ can be determined uniquely. Consider the subtensor $\mathcal{U}' = \mathcal{U}(y_1, \ldots, y_{j-1}, Y_j, y_{j+1}, \ldots, y_{d-1}, :) \in \mathbb{R}^{\overbrace{1 \times \ldots \times 1}^{j-1} \times r \times \overbrace{1 \times \ldots \times 1}^{d-j-1} \times n_d}$ which

includes rn_d entries. Having CP decomposition (2), each entry of $\mathcal{U}'$ results in one degree-1 polynomial in terms of the entries of $\mathbf{A}_d$ with coefficients in terms of the entries of $\mathbf{Q}_i$'s. Let the matrix $\mathbf{U}' \in \mathbb{R}^{r \times n_d}$ represent the rn_d entries of $\mathcal{U}'$. Moreover, define $\mathbf{C} = [\mathbf{c}_1|\ldots|\mathbf{c}_r] \in \mathbb{R}^{r \times r}$ where $\mathbf{c}_l = (\Pi_{i \in J} \mathbf{P}_i(1,l)) \mathbf{q}_j^l \in \mathbb{R}^{r \times 1}$ for $l = 1, \ldots, r$ and $\mathbf{q}_j^l \in \mathbb{R}^{r \times 1}$ is the l-th column of $\mathbf{Q}_j$.

Observe that CP decomposition (2) for the entries of $\mathcal{U}'$ can be written as $\mathbf{U}' = \mathbf{C}\mathbf{A}_d^\top$. Recall that $\mathbf{Q}_j$ is full rank, and therefore $\mathbf{q}_j^l$'s are linearly independent, $l = 1, \ldots, r$. Also, a system of equations with at least m linearly independent degree-1 polynomials in m variables does not have more than one solution. Hence, $\mathbf{c}_l$'s are also linearly independent for $l = 1, \ldots, r$, and therefore $\mathbf{C}$ is full rank. As a result, $\mathbf{A}_d$ can be determined uniquely. In the second step, similar to the first step, we can show that the rest of $\mathbf{A}_i$'s have at most one solution having one entry of $\mathbf{A}_d$ which has been already obtained.

Finally, we also claim that there exists at least one rank-r decomposition of $\mathcal{U}$ such that $\mathbf{P}_i = \mathbf{Q}_i$, $i = 1, \ldots, d-1$. We show this by induction on d. For $d = 2$, this is a result of Lemma 2. Induction hypothesis states that the claim holds for $d = k - 1$ and we need to show that it also holds for $d = k$. Since by merging dimension $k - 1$ and k for each of the rank-1 tensors of the corresponding CP decomposition and using induction hypothesis this step reduces to showing a rank-1 matrix can be decomposed to two vectors such that one component of one of them is given which is again a special case of Lemma 2 for rank-1 scenario. ∎

Assume that $\mathcal{S}$ denotes the set of all possible $\mathbf{A}_i$'s for $i = 1, \ldots, d-1$ given $\mathbf{A}_d$ without any polynomial constraint. Lemma 3 results in a pattern that characterizes exactly one rank-r decomposition among all rank-r decompositions, and therefore the dimension of $\mathcal{S}$ is equal to the number of unknowns, i.e., number of entries of $\mathbf{A}_i$'s for $i = 1, \ldots, d-1$ excluding those that are involved in the pattern $\mathbf{P}_i$'s in Lemma 3 which is $r(\sum_{i=1}^{d-1} n_i) - r^2 - r(d-2)$.

Lemma 4 *For almost every $\mathcal{U}$, the sampled tensor is finitely completable if and only if the maximum number of algebraically independent polynomials in $\mathcal{P}(\Omega)$ is equal to $r(\sum_{i=1}^{d-1} n_i) - r^2 - r(d-2)$.*

Proof The proof is omitted due to the similarity to the proof of Lemma 2 in (Ashraphijuo et al., 2016a) with the only difference that here the dimension is $r(\sum_{i=1}^{d-1} n_i) - r^2 - r(d-2)$ instead of $\left(\Pi_{i=1}^{j} n_i\right)\left(\Pi_{i=j+1}^{d} r_i\right) - \left(\sum_{i=j+1}^{d} r_i^2\right)$ which is the dimension of the core in Tucker decomposition. ∎

3.2 Constraint Tensor

In this section, we provide a procedure to construct a binary tensor $\breve{\Omega}$ based on Ω such that $\mathcal{P}(\breve{\Omega}) = \mathcal{P}(\Omega)$ and each polynomial can be represented by one d-way subtensor of $\breve{\Omega}$ which belongs to $\mathbb{R}^{n_1 \times n_2 \times \cdots \times n_{d-1} \times 1}$. Using $\breve{\Omega}$, we are able to recognize the observed entries that have been used to obtain the $\mathbf{A}_d$ in terms of the entries of $\mathbf{A}_1, \ldots, \mathbf{A}_{d-1}$, and we can study the algebraic independence of the polynomials in $\mathcal{P}(\Omega)$ which is directly related to finite completability through Lemma 4.

For each subtensor $\mathcal{Y}$ of the sampled tensor $\mathcal{U}$, let $N_\Omega(\mathcal{Y})$ denote the number of sampled entries in $\mathcal{Y}$. Specifically, consider any subtensor $\mathcal{Y} \in \mathbb{R}^{n_1 \times n_2 \times \cdots \times n_{d-1} \times 1}$ of the tensor $\mathcal{U}$. Then, since r of the polynomials have been used to obtain $\mathbf{A}_d$, $\mathcal{Y}$ contributes $N_\Omega(\mathcal{Y}) - r$ polynomial equations in terms of the entries of $\mathbf{A}_1, \ldots, \mathbf{A}_{d-1}$ among all $N_\Omega(\mathcal{U}) - rn_d$ polynomials in $\mathcal{P}(\Omega)$.

The sampled tensor $\mathcal{U}$ includes n_d subtensors that belong to $\mathbb{R}^{n_1 \times n_2 \times \cdots \times n_{d-1} \times 1}$ and let $\mathcal{Y}_i$ for $1 \leq i \leq n_d$ denote these n_d subtensors. Define a binary valued tensor $\breve{\mathcal{Y}}_i \in \mathbb{R}^{n_1 \times n_2 \times \cdots \times n_{d-1} \times k_i}$, where $k_i = N_\Omega(\mathcal{Y}_i) - r$ and its entries are described as the following. We can look at $\breve{\mathcal{Y}}_i$ as k_i tensors each belongs to $\mathbb{R}^{n_1 \times n_2 \times \cdots \times n_{d-1} \times 1}$. For each of the mentioned k_i tensors in $\breve{\mathcal{Y}}_i$ we set the entries corresponding to the r observed entries that are used to obtain $\mathbf{A}_d$ equal to 1. For each of the other k_i observed entries that have not been used to obtain $\mathbf{A}_d$, we pick one of the k_i tensors of $\breve{\mathcal{Y}}_i$ and set its corresponding entry (the same location as that specific observed entry) equal to 1 and set the rest of the entries equal to 0. In the case that $k_i = 0$ we simply ignore $\breve{\mathcal{Y}}_i$, i.e., $\breve{\mathcal{Y}}_i = \emptyset$

By putting together all n_d tensors in dimension d, we construct a binary valued tensor $\breve{\Omega} \in \mathbb{R}^{n_1 \times n_2 \times \cdots \times n_{d-1} \times K}$, where $K = \sum_{i=1}^{n_d} k_i = N_\Omega(\mathcal{U}) - r n_d$ and call it the **constraint tensor**. Observe that each subtensor of $\breve{\Omega}$ which belongs to $\mathbb{R}^{n_1 \times n_2 \times \cdots \times n_{d-1} \times 1}$ includes exactly $r + 1$ nonzero entries. In the following we show this procedure for an example.

Example 1 Consider an example in which $d = 3$ and $r = 2$ and $\mathcal{U} \in \mathbb{R}^{3 \times 3 \times 3}$. Assume that $\Omega(x, y, z) = 1$ if $(x, y, z) \in \mathcal{S}$ and $\Omega(x, y, z) = 0$ otherwise, where

$$\mathcal{S} = \{(1,1,1), (1,2,1), (2,3,1), (3,3,1), (1,1,2), (2,1,2), (3,2,2), (1,3,3), (3,2,3)\},$$

represents the set of observed entries. Hence, observed entries $(1,1,1), (1,2,1), (2,3,1), (3,3,1)$ belong to $\mathcal{Y}_1$, observed entries $(1,1,2), (2,1,2), (3,2,2)$ belong to $\mathcal{Y}_2$, and observed entries $(1,3,3)$, $(3,2,3)$ belong to $\mathcal{Y}_3$. As a result, $k_1 = 4 - 2 = 2$, $k_2 = 3 - 2 = 1$, and $k_3 = 2 - 2 = 0$. Hence, $\breve{\mathcal{Y}}_1 \in \mathbb{R}^{3 \times 3 \times 2}$, $\breve{\mathcal{Y}}_2 \in \mathbb{R}^{3 \times 3 \times 1}$, and $\breve{\mathcal{Y}}_3 = \emptyset$, and therefore the constraint tensor $\breve{\Omega}$ belongs to $\mathbb{R}^{3 \times 3 \times 3}$.

Also, assume that the entries that we use to obtain $\mathbf{A}_3$ in terms of the entries of $\mathbf{A}_1$ and $\mathbf{A}_2$ are $(2,3,1), (3,3,1), (1,1,2), (2,1,2), (1,3,3)$ and $(3,2,3)$. Note that $\breve{\mathcal{Y}}_1(2,3,1) = \breve{\mathcal{Y}}_1(2,3,2) = \breve{\mathcal{Y}}_1(3,3,1) = \breve{\mathcal{Y}}_1(3,3,2) = 1$ (correspond to entries of $\mathcal{Y}_1$ that have been used to obtain $\mathbf{A}_3$), and also for the two other observed entries we have $\breve{\mathcal{Y}}_1(1,1,1) = 1$ (correspond to $\mathcal{U}(1,1,1)$) and $\breve{\mathcal{Y}}_1(1,2,2) = 1$ (correspond to $\mathcal{U}(1,2,1)$) and the rest of the entries of $\breve{\mathcal{Y}}_1$ are equal to zero. Similarly, $\breve{\mathcal{Y}}_2(1,1,1) = \breve{\mathcal{Y}}_2(2,1,1) = \breve{\mathcal{Y}}_2(3,2,1) = 1$ and the rest of the entries of $\breve{\mathcal{Y}}_2$ are equal to zero.

Then, $\breve{\Omega}(x, y, z) = 1$ if $(x, y, z) \in \mathcal{S}'$ and $\breve{\Omega}(x, y, z) = 0$ otherwise, where

$$\breve{\mathcal{S}} = \{(1,1,1), (1,2,2), (2,3,1), (2,3,2), (3,3,1), (3,3,2), (1,1,3), (2,1,3), (3,2,3)\}.$$

Note that each subtensor of $\breve{\Omega}$ that belongs to $\mathbb{R}^{n_1 \times \cdots \times n_{d-1} \times 1}$ represents one of the polynomials in $\mathcal{P}(\Omega)$ besides showing the polynomials that have been used to obtain $\mathbf{A}_d$. More specifically, consider a subtensor of $\breve{\Omega}$ that belongs to $\mathbb{R}^{n_1 \times \cdots \times n_{d-1} \times 1}$ with $r + 1$ nonzero entries. Observe that exactly r of them correspond to the observed entries that have been used to obtain $\mathbf{A}_d$. Hence, this subtensor represents a polynomial after replacing entries of $\mathbf{A}_d$ by the expressions in terms of entries of $\mathbf{A}_1, \ldots, \mathbf{A}_{d-1}$, i.e., a polynomial in $\mathcal{P}(\Omega)$.

3.3 Algebraic Independence

In this section, we obtain the maximum number of algebraically independent polynomials in $\mathcal{P}(\breve{\Omega})$ in terms of the simple geometrical structure of nonzero entries of Ω, i.e., the locations of the sampled entries. On the other hand, Lemma 4 provides the required number of algebraically independent polynomials in $\mathcal{P}(\Omega)$ for finite completability. Hence, at the end of this section, we obtain the necessary and sufficient deterministic conditions on the sampling pattern for finite completability.

According to Lemma 3, as we consider one particular equivalence class some of the entries of $\mathbf{A}_i$'s are known, i.e., $\mathbf{P}_1, \ldots, \mathbf{P}_{d-1}$ in the statement of the lemma. Therefore, in order to find the number of variables (unknown entries of $\mathbf{A}_i$'s) in a set of polynomials, we should subtract the number of known entries in the corresponding pattern from the total number of involved entries. Also, recall that the sampled tensor is chosen generically from the corresponding manifold, and therefore according to Fact 2, the independency of the polynomials can be studied through studying the number of variables involved in each subset of them.

Definition 5 *Let $\breve{\Omega}' \in \mathbb{R}^{n_1 \times n_2 \times \cdots \times n_{d-1} \times t}$ be a subtensor of the constraint tensor $\breve{\Omega}$. Let $m_i(\breve{\Omega}')$ denote the number of nonzero rows of $\breve{\mathbf{\Omega}}'_{(i)}$. Also, let $\mathcal{P}(\breve{\Omega}')$ denote the set of polynomials that correspond to nonzero entries of $\breve{\Omega}'$.*

The following lemma gives an upper bound on the maximum number of algebraically independent polynomials in the set $\mathcal{P}(\breve{\Omega}')$ for an arbitrary subtensor $\breve{\Omega}' \in \mathbb{R}^{n_1 \times n_2 \times \cdots \times n_{d-1} \times t}$ of the constraint tensor. Note that $\mathcal{P}(\breve{\Omega}')$ includes exactly t polynomials as each subtensor belonging to $\mathbb{R}^{n_1 \times n_2 \times \cdots \times n_{d-1} \times 1}$ represents one polynomial.

Lemma 6 *Suppose that Assumption 1 holds. Consider an arbitrary subtensor $\breve{\Omega}' \in \mathbb{R}^{n_1 \times n_2 \times \cdots \times n_{d-1} \times t}$ of the constraint tensor $\breve{\Omega}$. The maximum number of algebraically independent polynomials in $\mathcal{P}(\breve{\Omega}')$ is at most*

$$r \left(\left(\sum_{i=1}^{d-1} m_i(\breve{\Omega}') \right) - min\left\{ max\left\{ m_1(\breve{\Omega}'), \ldots, m_{d-1}(\breve{\Omega}') \right\}, r \right\} - (d-2) \right). \tag{5}$$

Proof As a consequence of Fact 2, the maximum number of algebraically independent polynomials in a subset of polynomials of $\mathcal{P}(\breve{\Omega}')$ is at most equal to the total number of variables that are involved in the corresponding polynomials. Note that by observing the structure of (2) and Fact 1, the number of entries of $\mathbf{A}_i$ that are involved in the polynomials $\mathcal{P}(\breve{\Omega}')$ is equal to $rm_i(\breve{\Omega}')$, $i = 1, \ldots, d - 1$. Therefore, the total number of entries of $\mathbf{A}_1, \ldots, \mathbf{A}_{d-1}$ that are involved in the polynomials $\mathcal{P}(\breve{\Omega}')$ is equal to $r \left(\sum_{i=1}^{d-1} m_i(\breve{\Omega}') \right)$. However, some of the entries of $\mathbf{A}_1, \ldots, \mathbf{A}_{d-1}$ are known and depending on the equivalence class we should subtract them from the total number of involved entries.

For a fixed number j in Lemma 3, it is easily verified that the total number of variables (unknown entries) of $\mathbf{A}_1, \ldots, \mathbf{A}_{d-1}$ that are involved in the polynomials $\mathcal{P}(\breve{\Omega}')$ is equal to $r \sum_{i \in J} \left(m_i(\breve{\Omega}') - 1 \right)^+$ $+ r \left(m_j(\breve{\Omega}') - r \right)^+$, with $J = \{1, \ldots, d-1\} \backslash \{j\}$. Note that $\left(m_i(\breve{\Omega}') - 1 \right)^+ = m_i(\breve{\Omega}') - 1$, $\left(m_j(\breve{\Omega}') - r \right)^+ = m_j(\breve{\Omega}') - min\left\{ m_j(\breve{\Omega}'), r \right\}$. However, j is not a fixed number in general. Therefore, the maximum number of known entries of $\mathbf{A}_1, \ldots, \mathbf{A}_{d-1}$ that are involved in the polynomials $\mathcal{P}(\breve{\Omega}')$ is equal to $r \left(min\left\{ max\left\{ m_1(\breve{\Omega}'), \ldots, m_{d-1}(\breve{\Omega}') \right\}, r \right\} - (d-2) \right)$, which results that the number of variables of $\mathbf{A}_1, \ldots, \mathbf{A}_{d-1}$ that are involved in the polynomials $\mathcal{P}(\breve{\Omega}')$ is equal to (5). ∎

The set of polynomials corresponding to $\breve{\Omega}'$, i.e., $\mathcal{P}(\breve{\Omega}')$ is called minimally algebraically dependent if the polynomials in $\mathcal{P}(\breve{\Omega}')$ are algebraically dependent but polynomials in every of its

proper subsets are algebraically independent. The following lemma provides an important property about a set of minimally algebraically dependent $\mathcal{P}(\check{\Omega}')$. This lemma will be used later to derive the maximum number of algebraically independent polynomials in $\mathcal{P}(\check{\Omega}')$.

Lemma 7 *Suppose that Assumption 1 holds. Consider an arbitrary subtensor $\check{\Omega}' \in \mathbb{R}^{n_1 \times n_2 \times \cdots \times n_{d-1} \times t}$ of the constraint tensor $\check{\Omega}$. Assume that polynomials in $\mathcal{P}(\check{\Omega}')$ are minimally algebraically dependent. Then, the number of variables (unknown entries) of $\mathbf{A}_1, \ldots, \mathbf{A}_{d-1}$ that are involved in $\mathcal{P}(\check{\Omega}')$ is equal to $t - 1$.*

Proof Let the set of polynomials $\{p_1, , p_2 \ldots, p_t\}$ represent $\mathcal{P}(\check{\Omega}')$. Then, $\mathcal{P}(\check{\Omega}') \backslash p_i$ (all polynomials in $\mathcal{P}(\check{\Omega}')$ excluding p_i) is a set of algebraically independent polynomials, $i = 1, \ldots, t$. The number of involved variables in $\mathcal{P}(\check{\Omega}') \backslash p_i$ is at least equal to the number of polynomials, i.e., $t - 1$. Therefore, the number of involved variables in $\mathcal{P}(\check{\Omega}')$ is at least $t - 1$.

By contradiction, assume that the number of variables that are involved in the set of polynomials $\mathcal{P}(\check{\Omega}')$ is at least t. Then, we claim that for any subset of the polynomials of $\mathcal{P}(\check{\Omega}')$ the number of involved variables is at least equal to the number of polynomials in that subset. If the subset is equal to $\mathcal{P}(\check{\Omega}')$, the claim is just the assumption of contradiction. If the subset is a proper subset of $\mathcal{P}(\check{\Omega}')$, according to the assumption in the statement of Lemma, the polynomials are algebraically independent, and therefore, the claim holds. Hence, the polynomials in $\mathcal{P}(\check{\Omega}')$ are algebraically independent which contradicts the assumption. ∎

Given an arbitrary subtensor $\check{\Omega}' \in \mathbb{R}^{n_1 \times n_2 \times \cdots \times n_{d-1} \times t}$ of the constraint tensor $\check{\Omega}$, we are interested in obtaining the maximum number of algebraically independent polynomials in $\mathcal{P}(\check{\Omega}')$ based on the structure of nonzero entries of $\check{\Omega}'$. The next lemma can be used to characterize this number in terms of a simple geometric structure of nonzero entries of $\check{\Omega}'$.

Lemma 8 *Suppose that Assumption 1 holds. Consider an arbitrary subtensor $\check{\Omega}' \in \mathbb{R}^{n_1 \times n_2 \times \cdots \times n_{d-1} \times t}$ of the constraint tensor $\check{\Omega}$. The polynomials in $\mathcal{P}(\check{\Omega}')$ are algebraically independent if and only if for any $t' \in \{1, \ldots, t\}$ and any subtensor $\check{\Omega}'' \in \mathbb{R}^{n_1 \times n_2 \times \cdots \times n_{d-1} \times t'}$ of $\check{\Omega}'$ we have*

$$r\left(\left(\sum_{i=1}^{d-1} m_i(\check{\Omega}'')\right) - \min\left\{\max\left\{m_1(\check{\Omega}''), \ldots, m_{d-1}(\check{\Omega}'')\right\}, r\right\} - (d-2)\right) \geq t'. \quad (6)$$

Proof First, assume that all polynomials in $\mathcal{P}(\check{\Omega}')$ are algebraically independent. Also, by contradiction assume that there exists a subtensor $\check{\Omega}'' \in \mathbb{R}^{n_1 \times n_2 \times \cdots \times n_{d-1} \times t'}$ of $\check{\Omega}'$ that (6) does not hold for. Note that $\mathcal{P}(\check{\Omega}'')$ includes t' polynomials. On the other hand, according to Lemma 6, the maximum number of algebraically independent polynomials in $\mathcal{P}(\check{\Omega}'')$ is no greater than the LHS of (6), and therefore the polynomials in $\mathcal{P}(\check{\Omega}'')$ are not algebraically independent. Hence, the polynomials in $\mathcal{P}(\check{\Omega}')$ are not algebraically independent as well.

In order to prove the other side of the statement, assume that the polynomials in $\mathcal{P}(\check{\Omega}')$ are algebraically dependent. Hence, there exists a subset of the polynomials that are minimally algebraically dependent and let us denote it by $\mathcal{P}(\check{\Omega}'')$, where $\check{\Omega}'' \in \mathbb{R}^{n_1 \times n_2 \times \cdots \times n_{d-1} \times t'}$ is a subtensor of $\check{\Omega}'$. As stated in Lemma 7, the number of involved variables in polynomials in $\mathcal{P}(\check{\Omega}'')$ is equal to $t' - 1$. On the other hand, in the proof of Lemma 6, we showed that the number involved variables is at least

equal the LHS of (6). Therefore, the LHS of (6) is less than or equal to $t' - 1$ or equivalently

$$r\left(\left(\sum_{i=1}^{d-1} m_i(\breve{\Omega}'')\right) - \min\left\{\max\left\{m_1(\breve{\Omega}''), \ldots, m_{d-1}(\breve{\Omega}'')\right\}, r\right\} - (d-2)\right) < t'. \qquad (7)$$

∎

Finally, the following theorem characterizes the necessary and sufficient condition on $\breve{\Omega}$ for finite completability of the sampled tensor $\mathcal{U}$.

Theorem 9 *Suppose that Assumption* 1 *holds. For almost every* $\mathcal{U}$, *the sampled tensor* $\mathcal{U}$ *is finitely completable if and only if* $\breve{\Omega}$ *contains a subtensor* $\breve{\Omega}' \in \mathbb{R}^{n_1 \times n_2 \times \cdots \times n_{d-1} \times t}$ *such that (i)* $t = r(\sum_{i=1}^{d-1} n_i) - r^2 - r(d-2)$ *and (ii) for any* $t' \in \{1, \ldots, t\}$ *and any subtensor* $\breve{\Omega}'' \in \mathbb{R}^{n_1 \times n_2 \times \cdots \times n_{d-1} \times t'}$ *of* $\breve{\Omega}'$, (6) *holds.*

Proof Lemma 4 states that for almost every $\mathcal{U}$, there exist finitely many completions of the sampled tensor if and only if $\mathcal{P}(\breve{\Omega})$ includes $r(\sum_{i=1}^{d-1} n_i) - r^2 - r(d-2)$ algebraically independent polynomials. Moreover, according to Lemma 8, polynomials corresponding to a subtensor $\breve{\Omega}' \in \mathbb{R}^{n_1 \times n_2 \times \cdots \times n_{d-1} \times t}$ of the constraint tensor are algebraically independent if and only if condition (ii) of the statement of the Theorem holds. Therefore, for almost every $\mathcal{U}$, conditions (i) and (ii) hold if and only if the sampled tensor $\mathcal{U}$ is finitely completable. ∎

Note that the condition given in (6) is combinatorial in nature and hard to verify in practice. In the next section, we provide a lower bound on the number of samples so that both Assumption 1 and the combinatorial condition given in (6) hold true with high probability (not deterministically, i.e., with probability one anymore).

4. Probabilistic Conditions for Finite Completability

Assume that the entries of the tensor are sampled independently with probability p. In this section, we are interested in obtaining a condition in terms of the number of samples, i.e., the sampling probability, to ensure the finite completability of the sampled tensor with high probability. In Section 4.1, we apply the existing results on the Grassmannian manifold in (Pimentel-Alarcón et al., 2016d) on any of the unfoldings of the sampled tensor to derive the mentioned probabilistic condition. In Section 4.2, we obtain the conditions on the number of samples to ensure that conditions (i) and (ii) in the statement of Theorem 9 hold with high probability or in other words, to ensure the finite completability with high probability. For the notational simplicity in this section, we assume that $n_1 = n_2 = \cdots = n_d$, i.e., $\mathcal{U} \in \mathbb{R}^{n \times n \times \ldots \times n}$.

4.1 Unfolding Approach

In this section, we are interested in applying the existing analysis based on the Grassmannian manifold to obtain probabilistic conditions on the sampling pattern for finite completability with high probability. The following theorem restates Theorem 3 in (Pimentel-Alarcón et al., 2016d).

Theorem 10 *Consider an $n \times N$ matrix with the given rank k and let $0 < \epsilon < 1$ be given. Suppose $k \leq \frac{n}{6}$ and that each* **column** *of the sampled matrix is observed in at least l entries, distributed uniformly at random and independently across entries, where*

$$l > \max \left\{ 12 \, \log \left(\frac{n}{\epsilon} \right) + 12, 2k \right\}. \tag{8}$$

Also, assume that $k(n - k) \leq N$. Then, with probability at least $1 - \epsilon$, the sampled matrix will be finitely completable.

Observe that in the case of $1 < k < n - 1$, the assumption $k(n - k) \leq N$ results that $n < N$ which is very important to check when we apply this theorem. In order to use Theorem 10, we need the following lemma to obtain an upper bound on the rank of unfoldings of $\mathcal{U}$.

Lemma 11 *Consider an arbitrary nonempty $I \subset \{1, \ldots, d\}$ and recall that r denotes the CP rank of $\mathcal{U}$. Then, rank $\left(\widetilde{\mathbf{U}}_{(I)} \right) \leq r$.*

Proof In order to show rank $\left(\widetilde{\mathbf{U}}_{(I)} \right) \leq r$, we show the existence of a CP decomposition of $\widetilde{\mathbf{U}}_{(I)}$ with r components, i.e., we show that there exist $\mathbf{a}_i^l \in \mathbb{R}^{m_i}$ for $1 \leq i \leq 2$, $1 \leq l \leq r$, $m_1 \triangleq N_I$ and $m_2 \triangleq N_{\bar{I}}$ such that

$$\widetilde{\mathbf{U}}_{(I)} = \sum_{l=1}^r \mathbf{a}_1^l \otimes \mathbf{a}_2^l. \tag{9}$$

In order to do so, recall that since the CP rank of $\mathcal{U}$ is r, there exist $\mathbf{b}_i^l \in \mathbb{R}^{n_i}$ for $1 \leq i \leq d$ and $1 \leq l \leq r$ such that

$$\mathcal{U} = \sum_{l=1}^r \mathbf{b}_1^l \otimes \mathbf{b}_2^l \otimes \ldots \otimes \mathbf{b}_d^l. \tag{10}$$

Define $\mathcal{A}_1^l = \mathbf{b}_{i_1}^l \otimes \ldots \otimes \mathbf{b}_{i_{|I|}}^l$ and $\mathcal{A}_2^l = \mathbf{b}_{i_{|I|+1}}^l \otimes \ldots \otimes \mathbf{b}_{i_d}^l$ for $1 \leq l \leq l$, where $I = \{i_1, \ldots, i_{|I|}\}$, $\bar{I} = \{i_{|I|+1}, \ldots, i_d\}$. Let $\mathbf{a}_1^l$ and $\mathbf{a}_2^l$ denote the vectorizations of $\mathcal{A}_1^l$ and $\mathcal{A}_2^l$ with the same bijective mappings $\widetilde{M}_I : (x_{i_1}, \ldots, x_{i_{|I|}}) \to \{1, 2, \ldots, N_I\}$ and $\widetilde{M}_{\bar{I}} : (x_{i_{|I|+1}}, \ldots, x_{i_d}) \to \{1, 2, \ldots, \bar{N}_{\bar{I}}\}$ of the unfolding $\widetilde{\mathbf{U}}_{(I)}$. Hence, there exist $\mathbf{a}_i^l \in \mathbb{R}^{m_i}$ for $1 \leq i \leq 2$, $1 \leq l \leq r$ such that (9) holds. ∎

Remark 12 *Assume that $k \leq k' \leq \frac{n}{6}$, $l > \max \left\{ 12 \, \log \left(\frac{n}{\epsilon} \right) + 12, 2k' \right\}$ and $k'(n - k') \leq N$. Then, we have $l > \max \left\{ 12 \, \log \left(\frac{n}{\epsilon} \right) + 12, 2k \right\}$ since $k \leq k'$. Moreover, we have $k(n - k) \leq N$ since $k + k' < n$ which results $k(n - k) < k'(n - k')$.*

Lemma 13 *Let $I = \{i_1, \ldots, i_{|I|}\}$ be an arbitrary nonempty and proper subset of $\{1, \ldots, d\}$. Assume that $|I| < \frac{d}{2}$ and $r \leq \frac{n}{6}$, where r is the CP rank of the sampled tensor $\mathcal{U}$. Moreover, assume that each* **column** *of $\widetilde{\mathbf{U}}_{(I)}$ is observed in at least l entries, distributed uniformly at random and independently across entries, where*

$$l > \max \left\{ 12 \, \log \left(\frac{N_I r}{\epsilon} \right) + 12, 2r \right\}. \tag{11}$$

Then, with probability at least $1 - \epsilon$, the sampled tensor will be finitely completable.

14

Proof According to Lemma 11, $r_I \triangleq \text{rank}\left(\widetilde{\mathbf{U}}_{(I)}\right) \leq r$. Note that $N_I \leq n^{\frac{d-1}{2}} = \frac{n^{\frac{d+1}{2}}}{n} \leq \frac{N_{\bar{I}}}{n}$ which results that $r(N_I - r) \leq N_{\bar{I}}$. Furthermore, according to Remark 12, we have $r_I(N_I - r_I) \leq N_{\bar{I}}$ and also

$$l > \max\left\{12 \ \log\left(\frac{N_I r}{\epsilon}\right) + 12, 2r_I\right\}. \tag{12}$$

Therefore, according to Theorem 10, $\widetilde{\mathbf{U}}_{(I)}$ is finitely completable for an arbitrary value of r_I that belongs to $\{1, \ldots, r\}$ with probability at least $1 - \frac{\epsilon}{r}$. Hence, with probability at least $1 - \epsilon$, for all possible values of r_I, $\widetilde{\mathbf{U}}_{(I)}$ is finitely completable, i.e., $\widetilde{\mathbf{U}}_{(I)}$ is finitely completable. In order to complete to proof, it suffices to observe that finite completability of any of the unfoldings of $\mathcal{U}$ results the finite completability of $\mathcal{U}$. ∎

Remark 14 *In the case of $|I| > \frac{d}{2}$ in Lemma 13, we can simply consider the transpose of $\widetilde{\mathbf{U}}_{(I)}$ to have the similar results.*

Remark 15 *Lemma 13 requires*

$$n^{d-|I|} \max\left\{12 \ \log\left(\frac{n^{|I|}r}{\epsilon}\right) + 12, 2r\right\} \tag{13}$$

samples in total to ensure the finite completability of $\mathcal{U}$ with probability at least $1 - \epsilon$. Hence, the best bound on the total number of samples to ensure the finite completability with probability at least $1 - \epsilon$ will be obtained when $|I| = \lfloor\frac{d-1}{2}\rfloor$, which is

$$n^{\lceil\frac{d+1}{2}\rceil} \max\left\{12 \ \log\left(\frac{n^{\lfloor\frac{d-1}{2}\rfloor}r}{\epsilon}\right) + 12, 2r\right\}. \tag{14}$$

4.2 CP Approach

In this section, we present an approach based on the tensor CP decomposition instead of unfolding. Conditions (i) and (ii) in Theorem 9 ensure finite completability with probability one. Here, using combinatorial methods, we derive a lower bound on the number of sampled entries, i.e., the sampling probability, which ensures conditions (i) and (ii) in Theorem 9 hold with high probability. We first provide a few lemmas from our previous works. Lemma 16 below is Lemma 5 in (Ashraphijuo et al., 2017c), which will be used later.

Lemma 16 *Assume that $r' \leq \frac{n}{6}$ and also each column of $\mathbf{\Omega}_{(1)}$ (first matricization of Ω) includes at least l nonzero entries, where*

$$l > \max\left\{9 \ \log\left(\frac{n}{\epsilon}\right) + 3 \ \log\left(\frac{k}{\epsilon}\right) + 6, 2r'\right\}. \tag{15}$$

Let $\mathbf{\Omega}'_{(1)}$ be an arbitrary set of $n - r'$ columns of $\mathbf{\Omega}_{(1)}$. Then, with probability at least $1 - \frac{\epsilon}{k}$, every subset $\mathbf{\Omega}''_{(1)}$ of columns of $\mathbf{\Omega}'_{(1)}$ satisfies

$$m_1(\Omega'') - r' \geq t, \tag{16}$$

where t is the number of columns of $\mathbf{\Omega}''_{(1)}$ and $m_1(\Omega'')$ is the number of nonzero rows of $\mathbf{\Omega}''_{(1)}$.

The proof of the above lemma is based on a simple combinatorial idea to upper bound the number of possible patterns that do not satisfy the mentioned property.

Lemma 17 *Let $j \in \{1, 2, \ldots, d-1\}$ be a fixed number and $I = \{1, 2, \ldots, j\}$. Consider an arbitrary set $\widetilde{\mathbf{\Omega}}'_{(I)}$ of $n - r'$ columns of $\widetilde{\mathbf{\Omega}}_{(I)}$, where $r' \leq r \leq \frac{n}{6}$. Assume that $n > 200$, and also each column of $\widetilde{\mathbf{\Omega}}_{(I)}$ includes at least l nonzero entries, where*

$$l > \max \left\{ 27 \, \log \left(\frac{n}{\epsilon} \right) + 9 \, \log \left(\frac{2k}{\epsilon} \right) + 18, 6r' \right\}, \tag{17}$$

where $k \leq r$. Then, with probability at least $1 - \frac{\epsilon}{2k}$, each column of $\widetilde{\mathbf{\Omega}}'_{(I)}$ includes more than $l_0 \triangleq \max \left\{ 9 \, \log \left(\frac{n}{\epsilon} \right) + 3 \, \log \left(\frac{2k}{\epsilon} \right) + 6, 2r' \right\}$ observed entries of Ω with different values of the i-th coordinate, i.e., the i-th matricization of the tensor Ω' that corresponds to $\widetilde{\mathbf{\Omega}}'_{(I)}$ includes more than l_0 nonzero rows, $1 \leq i \leq j$.

Proof The proof is omitted due to the similarity to the proof for Lemma 9 in (Ashraphijuo and Wang, 2017). ∎

The following lemma is Lemma 8 in (Ashraphijuo et al., 2016a), which states that if the property in Lemma 16 holds for the sampling pattern Ω, it will be satisfied for $\check{\Omega}$ as well.

Lemma 18 *Let r' be a given nonnegative integer and $1 \leq i \leq j \leq d-1$ and $I = \{1, 2, \ldots, j\}$. Assume that there exists an $n^j \times (n - r')$ matrix $\widetilde{\mathbf{\Omega}}'_{(I)}$ composed of $n - r'$ columns of $\widetilde{\mathbf{\Omega}}_{(I)}$ such that each column of $\widetilde{\mathbf{\Omega}}'_{(I)}$ includes at least $r' + 1$ nonzero entries and satisfies the following property:*

- *Denote an $n^j \times t$ matrix (for any $1 \leq t \leq n - r'$) composed of any t columns of $\widetilde{\mathbf{\Omega}}'_{(I)}$ by $\widetilde{\mathbf{\Omega}}''_{(I)}$. Then*

$$m_i(\Omega'') - r' \geq t, \tag{18}$$

where $\widetilde{\mathbf{\Omega}}''_{(I)}$ is the unfolding of Ω'' corresponding to the set I.

Then, there exists an $n^j \times (n - r')$ matrix $\widetilde{\breve{\mathbf{\Omega}}}'_{(I)}$ such that: each column has exactly $r' + 1$ entries equal to one, and if $\widetilde{\breve{\mathbf{\Omega}}}'_{(I)}(x, y) = 1$ then we have $\widetilde{\mathbf{\Omega}}'_{(I)}(x, y) = 1$. Moreover, $\widetilde{\breve{\mathbf{\Omega}}}'_{(I)}$ satisfies the above-mentioned property.

The proof of the above lemma is based a novel generalization of Hall's theorem for bipartite graph. This generalization is shown by strong induction on the number of nodes.

Lemma 19 *Assume that $n > 200$, $1 \leq i \leq j \leq d-1$ and $I = \{1, 2, \ldots, j\}$. Consider r disjoint sets $\widetilde{\mathbf{\Omega}}'_{l_{(I)}}$, each with $n - r'_i$ columns of $\widetilde{\mathbf{\Omega}}_{(I)}$ for $1 \leq l \leq r$, where $r'_i \leq r \leq \frac{n}{6}$. Let $\widetilde{\mathbf{\Omega}}'_{(I)}$ denote the union of all r sets of columns $\widetilde{\mathbf{\Omega}}'_{l_{(I)}}$'s. Assume that each column of $\widetilde{\mathbf{\Omega}}_{(I)}$ includes at least l nonzero entries, where*

$$l > \max \left\{ 27 \, \log \left(\frac{n}{\epsilon} \right) + 9 \, \log \left(\frac{2rk}{\epsilon} \right) + 18, 6r'_i \right\}. \tag{19}$$

Then, there exists an $n^j \times r(n - r_i')$ matrix $\widetilde{\breve{\Omega}}'_{(I)}$ such that: each column has exactly $r_i' + 1$ entries equal to one, and if $\widetilde{\breve{\Omega}}'_{(I)}(x, y) = 1$ then we have $\widetilde{\Omega}'_{(I)}(x, y) = 1$ and also it satisfies the following property: with probability at least $1 - \frac{\epsilon}{k}$, every subset $\widetilde{\breve{\Omega}}''_{(I)}$ of columns of $\widetilde{\breve{\Omega}}'_{(I)}$ satisfies the following inequality

$$r\left(m_i(\breve{\Omega}'') - r_i'\right) \geq t, \tag{20}$$

where t is the number of columns of $\widetilde{\breve{\Omega}}''_{(I)}$ and $\breve{\Omega}''$ is the tensor corresponding to unfolding $\widetilde{\breve{\Omega}}''_{(I)}$.

Proof We first claim that with probability at least $1 - \frac{\epsilon}{kr}$, every subset $\widetilde{\Omega}''_{l_{(I)}}$ of columns of $\widetilde{\Omega}'_{l_{(I)}}$ satisfies

$$m_i(\Omega_l'') - r_i' \geq t, \tag{21}$$

where t is the number of columns of $\widetilde{\Omega}''_{l_{(I)}}$ and Ω_l'' is the tensor corresponding to unfolding $\widetilde{\Omega}''_{l_{(I)}}$. For simplicity we denote the above-mentioned property by Property I. According to Lemma 17, with probability at least $1 - \frac{\epsilon}{2kr}$, the i-th matricization of the tensor Ω_l'' includes more than $\max\left\{9 \log\left(\frac{n}{\epsilon}\right) + 3 \log\left(\frac{2kr}{\epsilon}\right) + 6, 2r_i'\right\}$ nonzero rows, $1 \leq i \leq j$, and we denote this property by Property II. On the other hand, given that Property II holds for Ω_l'' and according to Lemma 16, with probability at least $1 - \frac{\epsilon}{2kr}$, Property I holds for Ω_l'' as well. Hence, with probability at least $1 - \frac{\epsilon}{kr}$, Property I holds for Ω_l'', which completes the proof our earlier claim.

Consequently, according to Lemma 18, with probability at least $1 - \frac{\epsilon}{kr}$, there exists an $n^j \times (n - r_i')$ matrix $\widetilde{\breve{\Omega}}'_{l_{(I)}}$ such that: each column has exactly $r_i' + 1$ entries equal to one, and if $\widetilde{\breve{\Omega}}'_{l_{(I)}}(x, y) = 1$ then we have $\widetilde{\Omega}'_{l_{(I)}}(x, y) = 1$ and also $\breve{\Omega}_l'$ satisfies Property I. Finally define $\widetilde{\breve{\Omega}}'_{(I)} \triangleq \left[\widetilde{\breve{\Omega}}'_{1_{(I)}} | \widetilde{\breve{\Omega}}'_{2_{(I)}} | \ldots | \widetilde{\breve{\Omega}}'_{r_{(I)}}\right]$. Since each $\breve{\Omega}_l'$ satisfies Property I with probability at least $1 - \frac{\epsilon}{kr}$, all $\breve{\Omega}_l'$'s satisfy Property I with probability at least $1 - \frac{\epsilon}{k}$, simultaneously. Consider an arbitrary subset $\widetilde{\breve{\Omega}}''_{(I)}$ of columns of $\widetilde{\breve{\Omega}}'_{(I)}$. Let $\widetilde{\breve{\Omega}}''_{l_{(I)}}$ denote those columns of $\widetilde{\breve{\Omega}}''_{(I)}$ that belong to $\widetilde{\breve{\Omega}}'_{l_{(I)}}$ and define t_l as the number of columns of $\widetilde{\breve{\Omega}}''_{l_{(I)}}$, $1 \leq l \leq r$, and define t as the number of columns of $\widetilde{\breve{\Omega}}''_{(I)}$. Without loss of generality, assume that $t_1 \leq t_2 \leq \ldots \leq t_r$. Also, assume that all $\breve{\Omega}_l'$'s satisfy Property I. Hence, we have

$$t = \sum_{l=1}^{r} t_l \leq rt_r \leq r\left(m_i(\breve{\Omega}_r'') - r_i'\right) \leq r\left(m_i(\breve{\Omega}'') - r_i'\right). \tag{22}$$

∎

Theorem 20 *Assume that $d > 2$, $n > \max\{200, r(d - 2)\}$, $r \leq \frac{n}{6}$ and $I = \{1, 2, \ldots, d - 2\}$. Assume that each column of $\widetilde{\Omega}_{(I)}$ includes at least l nonzero entries, where*

$$l > \max\left\{27 \log\left(\frac{n}{\epsilon}\right) + 9 \log\left(\frac{2r(d - 2)}{\epsilon}\right) + 18, 6r\right\}. \tag{23}$$

17

Then, with probability at least $1 - \epsilon$, for almost every $\mathcal{U} \in \mathbb{R}^{\overbrace{n \times \cdots \times n}^{d}}$, there exist only finitely many completions of the sampled tensor $\mathcal{U}$ with CP rank r.

Proof Define the $(d-1)$-way tensor $\mathcal{U}' \in \mathbb{R}^{\overbrace{n \times \cdots \times n}^{d-2} \times n^2}$ which is obtained through merging the $(d-1)$-th and d-th dimensions of the tensor $\mathcal{U}$. Observe that the finiteness of the number of completions of the tensor $\mathcal{U}'$ of rank r ensures the finiteness of the number of completions of the tensor $\mathcal{U}$ of rank r. For notational simplicity, let Ω and $\breve{\Omega}$ denote the $(d-1)$-way sampling pattern and constraint tensors corresponding to $\mathcal{U}'$, respectively. In order to complete the proof it suffices to show with probability at least $1 - \epsilon$, conditions (i) and (ii) in Theorem 9 hold for this modified $(d-1)$-way tensor.

Now, we apply Lemma 19 for each of the numbers $r'_1 = 1, \ldots, r'_{d-3} = 1, r'_{d-2} = r$. Also, note that since $n > r(d-2)$ we conclude $n^2 > r(n-r) + (d-3)r(n-1)$, and therefore $\widetilde{\Omega}_{(I)}$ includes more than $r(n-r) + (d-3)r(n-1)$ columns. According to Lemma 19, there exist $\breve{\Omega}'_i$ for $1 \le i \le d-2$ such that: (i) each column of $\widetilde{\Omega}'_{i_{(I)}}$ includes $r'_i + 1$ nonzero entries for $1 \le i \le d-2$, and if $\widetilde{\Omega}'_{i_{(I)}}(x,y) = 1$ then we have $\widetilde{\Omega}'_{i_{(I)}}(x,y) = 1$, (ii) $\widetilde{\Omega}'_{i_{(I)}}$ includes $r(n-1)$ and $r(n-r)$ columns for $1 \le i \le d-3$ and $i = d-2$, respectively, (iii) with probability at least $1 - \frac{\epsilon}{d-2}$, every subset $\widetilde{\Omega}''_{i_{(I)}}$ of columns of $\widetilde{\Omega}'_{i_{(I)}}$ satisfies (20) for $r'_1 = 1, \ldots, r'_{d-3} = 1, r'_{d-2} = r$.

Recall that each column of $\widetilde{\Omega}_{(I)}$ includes $r + 1$ nonzero entries, and therefore for $1 \le i \le d-3$ that we have $r'_i + 1 = 2$, the column of $\widetilde{\Omega}_{(I)}$ corresponding to an column of $\widetilde{\Omega}'_{i_{(I)}}$ has $r - 1$ more nonzero entries.

Observe that $\max\left\{9 \log\left(\frac{n}{\epsilon}\right) + 3 \log\left(\frac{2kr}{\epsilon}\right) + 6, 2r\right\} \ge 2r \ge (r-1) + 2$. According to Lemma 17 and given (23), for each column of $\widetilde{\Omega}'_{i_{(I)}}$ there exists another $r - 1$ zero entries (x_s, y_s) for $s \in \{1, \ldots, r-1\}$ in different rows of the $(d-2)$-th matricization of $\breve{\Omega}'_i$ from the two nonzero entries such that $\widetilde{\Omega}'_{i_{(I)}}(x_s, y_s) = 1, i = 1, \ldots, d-3$. Hence, for each column of $\widetilde{\Omega}'_{i_{(I)}}$, we substitute it with the column of $\widetilde{\Omega}_{(I)}$ that the value of such $r - 1$ entries (in different rows of the $(d-2)$-th matricization of $\breve{\Omega}'_i$) is 1 instead of 0, $i = 1, \ldots, d-3$. Therefore, each column of $\breve{\widetilde{\Omega}}'_{i_{(I)}}$ includes exactly $r + 1$ nonzero entries for $1 \le i \le d-3$ such that if $\breve{\widetilde{\Omega}}'_{i_{(I)}}(x,y) = 1$ then we have $\widetilde{\Omega}'_{i_{(I)}}(x,y) = 1$.

Let $\breve{\widetilde{\Omega}}'_{(I)} = \left[\breve{\widetilde{\Omega}}'_{1_{(I)}} | \ldots | \breve{\widetilde{\Omega}}'_{d-2_{(I)}}\right]$, which includes $r(n-r) + (d-3)r(n-1)$ columns. Therefore, $\breve{\Omega}' \in \mathbb{R}^{\overbrace{n \times \cdots \times n}^{d-2} \times t}$ is a subtensor of the constraint tensor such that $t = r(\sum_{i=1}^{d-2} n) - r^2 - r(d-3)$ and also and with probability at least $1 - \epsilon$, every subset $\breve{\widetilde{\Omega}}''_{i_{(I)}}$ of columns of $\breve{\widetilde{\Omega}}'_{i_{(I)}}$ satisfies (20) for $r'_1 = 1, \ldots, r'_{d-3} = 1, r'_{d-2} = r$, simultaneously for $i = 1, \ldots, d-2$.

Consider an arbitrary subset $\widetilde{\Omega}''_{(I)}$ of columns of $\breve{\widetilde{\Omega}}'_{(I)}$. Let $\widetilde{\Omega}''_{i_{(I)}}$ denote those columns of $\widetilde{\Omega}''_{(I)}$ that belong to $\breve{\widetilde{\Omega}}'_{i_{(I)}}$ and define t_i as the number of columns of $\widetilde{\Omega}''_{i_{(I)}}$, $1 \le i \le d-2$, and define t

18

as the number of columns of $\widetilde{\widetilde{\Omega}}''_{(I)}$. Also, assume that all $\check{\Omega}'_l$'s satisfy (20) for $r'_1 = 1, \ldots, r'_{d-3} = 1, r'_{d-2} = r$. Then, we have the following two scenarios:

(i) $t_{d-2} = 0$: Hence, we have $t = \sum_{i=1}^{d-3} t_i$. Moreover, we have

$$t_i \leq r \left(m_i(\check{\Omega}''_i) - 1 \right)^+ \leq r \left(m_i(\check{\Omega}'') - 1 \right)^+ = r \left(m_i(\check{\Omega}'') - 1 \right), \tag{24}$$

for $1 \leq i \leq d - 3$. Recall that each column of $\widetilde{\Omega}'_{i_{(I)}}$ includes at least r nonzero entries in different rows of the $(d-2)$-th matricization of $\check{\Omega}'_i$ for $1 \leq i \leq d-3$. On the other hand, since $\widetilde{\widetilde{\Omega}}''_{(I)}$ includes at least one column of $\left[\widetilde{\Omega}'_{1_{(I)}} | \ldots | \widetilde{\Omega}'_{d-3_{(I)}} \right]$ (recall that $t_{d-2} = 0$), we have

$$r \leq m_{d-2}(\check{\Omega}''), \tag{25}$$

which also results that $\min \left\{ \max \left\{ m_1(\check{\Omega}''), \ldots, m_{d-2}(\check{\Omega}'') \right\}, r \right\} = r$.

Therefore, having (24) and (25), we conclude

$$t = \sum_{i=1}^{d-3} t_i \leq \sum_{i=1}^{d-3} r \left(m_i(\check{\Omega}'') - 1 \right) \leq \sum_{i=1}^{d-3} r \left(m_i(\check{\Omega}'') - 1 \right) + r \left(m_{d-2}(\check{\Omega}'') - r \right) \tag{26}$$

$$= r \left(\sum_{i=1}^{d-2} m_i(\check{\Omega}'') \right) - r^2 - r(d-3)$$

$$= r \left(\left(\sum_{i=1}^{d-2} m_i(\check{\Omega}'') \right) - \min \left\{ \max \left\{ m_1(\check{\Omega}''), \ldots, m_{d-2}(\check{\Omega}'') \right\}, r \right\} - (d-3) \right).$$

(ii) $t_{d-2} > 0$: Hence, we have

$$t_{d-2} \leq r \left(m_{d-2}(\check{\Omega}''_{d-2}) - r \right) \leq r \left(m_{d-2}(\check{\Omega}'') - r \right), \tag{27}$$

which also results that $\min \left\{ \max \left\{ m_1(\check{\Omega}''), \ldots, m_{d-2}(\check{\Omega}'') \right\}, r \right\} = r$. Moreover, similar to scenario (i), (24) holds. Therefore, having (24) and (27), we conclude

$$t = \sum_{i=1}^{d-2} t_i \leq \sum_{i=1}^{d-3} r \left(m_i(\check{\Omega}'') - 1 \right) + r \left(m_{d-2}(\check{\Omega}'') - r \right) = r \left(\sum_{i=1}^{d-2} m_i(\check{\Omega}'') \right) - r^2 - r(d-3) \tag{28}$$

$$= r \left(\left(\sum_{i=1}^{d-2} m_i(\check{\Omega}'') \right) - \min \left\{ \max \left\{ m_1(\check{\Omega}''), \ldots, m_{d-2}(\check{\Omega}'') \right\}, r \right\} - (d-3) \right).$$

∎

Remark 21 *Observe that each row of the d-matricization of $\mathcal{U}$ consists of exactly n columns of $\widetilde{\Omega}_{(I)}$, where $I = \{1, 2, \ldots, d - 2\}$. Hence, given (23), Assumption 1 holds with probability one since Assumption 1 only requires r sampled entries at each row of the d-matricization of $\mathcal{U}$. Moreover, note that in the above theorem we showed that given (23), constraint (6) holds with high probability and this ensures finite completability with high probability (since Assumption 1 holds with probability one).*

Note that for the general values of $n_1, \ldots, n_d$ the same proof will still work, but instead of the assumption $n > \max\{200, r(d-2)\}$, we need another assumption in terms of $n_1, \ldots, n_d$ to ensure that the corresponding unfolding has enough number of columns.

Remark 22 *A tensor $\mathcal{U}$ that satisfies the properties in the statement of Theorem 20 requires*

$$n^2 \max \left\{ 27 \, \log \left(\frac{n}{\epsilon} \right) + 9 \, \log \left(\frac{2r(d-2)}{\epsilon} \right) + 18, 6r \right\} \tag{29}$$

samples to be finitely completable with probability at least $1 - \epsilon$, which is lower than the number of samples required by the unfolding approach given in (14) *by orders of magnitude.*

The following lemma is taken from (Ashraphijuo et al., 2016a) and is used in Lemma 24 to derive a lower bound on the sampling probability that results (23) with high probability.

Lemma 23 *Consider a vector with n entries where each entry is observed with probability p independently from the other entries. If $p > p' = \frac{k}{n} + \frac{1}{\sqrt[4]{n}}$, then with probability at least $\left(1 - \exp(-\frac{\sqrt{n}}{2}) \right)$, more than k entries are observed.*

The proof of the above lemma is simply based on Azuma's inequality.

Lemma 24 *Assume that $d > 2$, $n > \max\{200, r(d-2)\}$ and $r \leq \frac{n}{6}$. Moreover, assume that the sampling probability satisfies*

$$p > \frac{1}{n^{d-2}} \max \left\{ 27 \, \log \left(\frac{n}{\epsilon} \right) + 9 \, \log \left(\frac{2r(d-2)}{\epsilon} \right) + 18, 6r \right\} + \frac{1}{\sqrt[4]{n^{d-2}}}. \tag{30}$$

Then, with probability at least $(1 - \epsilon) \left(1 - \exp(-\frac{\sqrt{n^{d-2}}}{2}) \right)^{n^2}$, $\mathcal{U}$ is finitely completable.

Proof According to Lemma 23, (30) results that each column of $\widetilde{\Omega}_{(I)}$ includes at least l nonzero entries, where $I = \{1, 2, \ldots, d-2\}$ and l satisfies (23) with probability at least $\left(1 - \exp(-\frac{\sqrt{n^{d-2}}}{2}) \right)$. Therefore, with probability at least $\left(1 - \exp(-\frac{\sqrt{n^{d-2}}}{2}) \right)^{n^2}$, all n^2 columns of $\widetilde{\Omega}_{(I)}$ satisfy (23). Hence, according to Theorem 20, with probability at least $(1 - \epsilon) \left(1 - \exp(-\frac{\sqrt{n^{d-2}}}{2}) \right)^{n^2}$, $\mathcal{U}$ is finitely completable. ∎

5. Deterministic and Probabilistic Conditions for Unique Completability

In this section, we are interested in characterizing the deterministic and probabilistic conditions on the sampling pattern for unique completability. In previous sections we characterized the corresponding conditions for finite completability in Theorem 9 and Theorem 20. However, for matrix and tensor completion problems, finite completability does not necessarily imply unique completability (Ashraphijuo et al., 2016a). In this section, we add some additional mild restrictions on

Ω in the statement of Theorem 9 to ensure unique completability (deterministic) and also increase the number of samples given in the statement of Theorem 20 mildly to ensure unique completability with high probability (probabilistic). As the first step of this procedure, we use the following lemma for minimally algebraically dependent polynomials to obtain the variables involved in these polynomials uniquely. Hence, by obtaining all entries of the CP decomposition of the sampled tensor $\mathcal{U}$ we can show the uniqueness of $\mathcal{U}$.

Lemma 25 *Suppose that Assumption 1 holds. Let $\breve{\Omega}' \in \mathbb{R}^{n_1 \times n_2 \times \ldots \times n_{d-1} \times t}$ be an arbitrary subtensor of the constraint tensor $\breve{\Omega}$. Assume that polynomials in $\mathcal{P}(\breve{\Omega}')$ are minimally algebraically dependent. Then, all variables (unknown entries) of $\mathbf{A}_1, \ldots, \mathbf{A}_{d-1}$ that are involved in $\mathcal{P}(\breve{\Omega}')$ can be determined uniquely.*

Proof According to Lemma 7, the number of involved variables in polynomials in $\mathcal{P}(\breve{\Omega}') = \{p_1, p_2, \ldots, p_t\}$ is $t - 1$ and are denoted by $\{x_1, \ldots, x_{t-1}\}$. Moreover, as mentioned in the proof of Lemma 7, $\mathcal{P}(\breve{\Omega}') \backslash p_i$ is a set of algebraically independent polynomials and the number of involved variables is $t - 1$, $i = 1, \ldots, t$. Consider an arbitrary variable x_1 that is involved in polynomials in $\mathcal{P}(\breve{\Omega}')$ and without loss of generality, assume that x_1 is involved in p_1.

On the other hand, as mentioned all variables $\{x_1, \ldots, x_{t-1}\}$ are involved in algebraically independent polynomials in $\mathcal{P}(\breve{\Omega}') \backslash p_1$. Hence, tuple $(x_1, \ldots, x_{t-1})$ can be determined finitely. Given that one of these finitely many tuples $(x_1, \ldots, x_{t-1})$ satisfy polynomial equation p_1, with probability one there does not exist another tuple $(x_1, \ldots, x_{t-1})$ among them that satisfies polynomial equation p_1. This is because with probability zero a tuple satisfy a polynomial equation in which the coefficients are chosen generically, and also the fact that the number of such tuples is finite. ∎

Condition (i) in Theorem 26 results in $r(\sum_{i=1}^{d-1} n_i) - r^2 - r(d-2)$ algebraically independent polynomials in terms of the entries of $\mathbf{A}_1, \ldots, \mathbf{A}_{d-1}$, i.e., results in finite completability. Hence, adding a single polynomial corresponding to any observed entry to these $r(\sum_{i=1}^{d-1} n_i) - r^2 - r(d-2)$ algebraically independent polynomials results in a set of algebraically dependent polynomials. Then, according to Lemma 25 a subset of the entries of $\mathbf{A}_1, \ldots, \mathbf{A}_{d-1}$ can be determined uniquely and these additional polynomials are captured in the structure of condition (ii) such that all entries of CP decomposition can be determined uniquely.

Theorem 26 *Suppose that Assumption 1 holds. Also, assume that there exist disjoint subtensors $\breve{\Omega}' \in \mathbb{R}^{n_1 \times n_2 \times \cdots \times n_{d-1} \times t}$ and $\breve{\Omega}'^i \in \mathbb{R}^{n_1 \times n_2 \times \cdots \times n_{d-1} \times t_i}$ (for $1 \leq i \leq 2d - 2$) of the constraint tensor such that the following conditions hold:*

(i) $t = r(\sum_{i=1}^{d-1} n_i) - r^2 - r(d-2)$ and for any $t' \in \{1, \ldots, t\}$ and any subtensor $\breve{\Omega}'' \in \mathbb{R}^{n_1 \times n_2 \times \cdots \times n_{d-1} \times t'}$ of $\breve{\Omega}'$, (6) holds.

(ii) for $i \in \{1, \ldots, d-1\}$ we have $t_i = n_i - 1$ and for $i \in \{d, \ldots, 2d-2\}$ we have $t_i = n_{i-d+1} - r$. Also, for any $t'_i \in \{1, \ldots, t_i\}$ and any subtensor $\breve{\Omega}''^i \in \mathbb{R}^{n_1 \times n_2 \times \cdots \times n_{d-1} \times t'_i}$ of the tensor $\breve{\Omega}'^i$, the following inequalities hold

$$m_i(\breve{\Omega}''^i) - 1 \geq t'_i, \qquad for\ i \in \{1, \ldots, d-1\}, \tag{31}$$

and

$$m_{i-d+1}(\breve{\Omega}''^i) - r \geq t'_i, \qquad for\ i \in \{d, \ldots, 2d-2\}. \tag{32}$$

Then, for almost every $\mathcal{U}$, there exists only a unique tensor that fits in the sampled tensor $\mathcal{U}$, and has CP rank r.

Proof As we showed in the proof of Theorem 9, $\mathcal{P}(\breve{\Omega}')$ includes $t = r(\sum_{i=1}^{d-1} n_i) - r^2 - r(d-2)$ algebraically independent polynomials which results the finite completability of the sampled tensor $\mathcal{U}$. Let $\{p_1, \ldots, p_t\}$ denote these t algebraically independent polynomials in $\mathcal{P}(\breve{\Omega}')$. Now, having $\{p_1, \ldots, p_t\}$ and $\mathcal{P}(\breve{\Omega}'^i)$ for $1 \leq i \leq 2d-2$, and using Lemma 25 several times, we show the unique completability. Recall that t is the number of total variables among the polynomials, and therefore union of any polynomial p_0 and $\{p_1, \ldots, p_t\}$ is a set of algebraically dependent polynomials. Hence, there exists a set of polynomials $\mathcal{P}(\breve{\Omega}'')$ such that $\mathcal{P}(\breve{\Omega}'') \subset \{p_1, \ldots, p_t\}$ and also polynomials in $\mathcal{P}(\breve{\Omega}'') \cup p_0$ are minimally algebraically dependent polynomials. Therefore, according to Lemma 25, all variables involved in the polynomials $\mathcal{P}(\breve{\Omega}'') \cup p_0$ can be determined uniquely, and consequently, all variables involved in p_0 can be determined uniquely.

We can repeat the above procedure for any polynomial $p_0 \in \mathcal{P}(\breve{\Omega}'^i)$ to determine the involved variables uniquely with the help of $\{p_1, \ldots, p_t\}$, $i = 1, \ldots, 2d-2$. Hence, for any polynomial $p_0 \in \mathcal{P}(\breve{\Omega}'^i)$ or $p_0 \in \mathcal{P}(\breve{\Omega}'^{i+d-1})$, we obtain r degree-1 polynomials in terms of the entries of $\mathbf{A}_i$ but some of the entries of CP decomposition are elements of the $\mathbf{Q}_i$ matrices (in the statement of Lemma 3), $i = 1, \ldots, d-1$. In order to complete the proof, we need to show that condition (ii) with the above procedure using $\{p_1, \ldots, p_t\}$ results in obtaining all variables uniquely. In particular, we show that repeating the described procedure for any of the polynomials in $\mathcal{P}(\breve{\Omega}'^i)$ and $\mathcal{P}(\breve{\Omega}'^{i+d-1})$ result in obtaining all variables of the i-th element of CP decomposition uniquely.

According to Lemma 3, we have the following two scenarios for any $i \in \{1, \ldots, d-1\}$:

(i) $\mathbf{Q}_i \in \mathbb{R}^{r \times r}$: In this case, condition (ii) for $\breve{\Omega}''^{i+d-1}$ and for any $t'_{i+d-1} \in \{1, \ldots, n_i - r\}$ and any subtensor $\breve{\Omega}''^{i+d-1} \in \mathbb{R}^{n_1 \times n_2 \times \cdots \times n_{d-1} \times t'_{i+d-1}}$ of the tensor $\breve{\Omega}'^{i+d-1}$ results

$$rm_i(\breve{\Omega}''^{i+d-1}) - r^2 \geq rt'_{i+d-1}. \tag{33}$$

Note that rt'_{i+d-1} is the number of polynomials from the above mentioned procedure corresponding to $\breve{\Omega}''^{i+d-1}$ and $rm_i(\breve{\Omega}''^{i+d-1})$ denotes the number of involved entries of $\mathbf{A}_i$ in these polynomials, and therefore $rm_i(\breve{\Omega}''^{i+d-1}) - r^2$ is the number of involved variables of $\mathbf{A}_i$ in these polynomials. As a result, according to Fact 2, given $\mathcal{P}(\breve{\Omega}'^{i+d-1})$ and $\{p_1, \ldots, p_t\}$, the mentioned procedure results in $rn_i - r^2$ algebraically independent degree-1 polynomials in terms of the unknown entries of $\mathbf{A}_i$. Therefore, $\mathbf{A}_i$ can be determined uniquely.

(ii) $\mathbf{Q}_i \in \mathbb{R}^{1 \times r}$: Similar to scenario (i), condition (ii) for $\breve{\Omega}''^i$ and for any $t'_i \in \{1, \ldots, n_i\}$ and any subtensor $\breve{\Omega}''^i \in \mathbb{R}^{n_1 \times n_2 \times \cdots \times n_{d-1} \times t'_i}$ of the tensor $\breve{\Omega}'^i$ results

$$rm_i(\breve{\Omega}''^i) - r \geq rt'_i, \tag{34}$$

and therefore similar to the previous scenario, $\mathbf{A}_i$ can be determined uniquely. ■

In Theorem 26, we obtained the deterministic condition on the sampling pattern for unique completability. Note that Condition (i) in Theorem 26 is the same condition for finite completability.

In the remainder of this section, we are interested in characterizing the probabilistic conditions on the number of samples to ensure unique completability with high probability. For the sake

of simplicity, as in Section 4 we consider the sampled tensor $\mathcal{U} \in \mathbb{R}^{\overbrace{n \times \ldots \times n}^{d}}$. Recall that for the general values of $n_1, \ldots, n_d$ the same proof will still work, but instead of assumption $n > \max\{200, (r+2)(d-2)\}$, we need another assumption in terms of $n_1, \ldots, n_d$.

Theorem 27 *Assume that $d > 2$, $n > \max\{200, (r+2)(d-2)\}$, $r \leq \frac{n}{6}$ and $I = \{1, 2, \ldots, d-2\}$. Assume that each column of $\widetilde{\mathbf{\Omega}}_{(I)}$ includes at least l nonzero entries, where*

$$l > \max\left\{27 \, \log\left(\frac{2n}{\epsilon}\right) + 9 \, \log\left(\frac{8r(d-2)}{\epsilon}\right) + 18, 6r\right\}. \tag{35}$$

Then, with probability at least $1 - \epsilon$, for almost every $\mathcal{U} \in \mathbb{R}^{\overbrace{n \times \ldots \times n}^{d}}$, there exist only one completion of the sampled tensor $\mathcal{U}$ with CP rank r.

Proof Similar to the proof of Theorem 20, define the $(d-1)$-way tensor $\mathcal{U}' \in \mathbb{R}^{\overbrace{n \times \ldots \times n}^{d-2} \times n^2}$ which is obtained through merging the $(d-1)$-th and d-th dimensions of the tensor $\mathcal{U}$ and recall that the finiteness of the number of completions of the tensor $\mathcal{U}'$ of rank r ensures the finiteness of the number of completions of the tensor $\mathcal{U}$ with rank r. Similarly, for simplicity, assume that Ω and $\check{\Omega}$ denote the $(d-1)$-way sampling pattern and constraint tensors corresponding to $\mathcal{U}'$, respectively. Note that since $n > (r+2)(d-2)$, we conclude $n^2 > r(n-r) + (d-3)r(n-1) + 2(d-2)$, and therefore $\widetilde{\mathbf{\Omega}}_{(I)}$ includes more than $r(n-r) + (d-3)r(n-1) + 2n(d-2)$ columns. According to the proof of Theorem 20, considering $r(n-r) + (d-3)r(n-1)$ arbitrary columns of $\widetilde{\mathbf{\Omega}}_{(I)}$ results in existence of $\check{\Omega}'$ such that condition (i) holds with probability at least $1 - \frac{\epsilon}{2}$. Also, there exist at least $2n(d-2)$ columns other than these $r(n-r) + (d-3)r(n-1)$ columns.

Consider $n-1$ arbitrary columns of $\widetilde{\mathbf{\Omega}}_{(I)}$. By setting $r = 1$ in the statement of Lemma 19, these $n-1$ columns result in $\check{\Omega}''^i$ with $n-1$ columns such that with probability at least $1 - \frac{\epsilon}{4r(d-2)}$, (31) holds. Similarly, consider $n-r$ arbitrary columns of $\widetilde{\mathbf{\Omega}}_{(I)}$. Then, there exists $\check{\Omega}''^{i+d-2}$ such that with probability at least $1 - \frac{\epsilon}{4r(d-2)}$, (32) holds. Hence, condition (ii) holds (for all $i \in \{1, \ldots, 2d-4\}$) with probability at least $1 - \frac{\epsilon}{2r}$. Therefore, conditions (i) and (ii) hold with probability at least $1 - \left(\frac{\epsilon}{2r} + \frac{\epsilon}{2}\right) \geq 1 - \epsilon$. ∎

Remark 28 *A tensor $\mathcal{U}$ that satisfies the properties in the statement of Theorem 27 requires*

$$n^2 \max\left\{27 \, \log\left(\frac{2n}{\epsilon}\right) + 9 \, \log\left(\frac{8r(d-2)}{\epsilon}\right) + 18, 6r\right\} \tag{36}$$

samples to be uniquely completable with probability at least $1 - \epsilon$, which is orders-of-magnitude lower than the number of samples required by the unfolding approach given in (14). Note that the number of samples given in Theorem 3 of (Pimentel-Alarcón et al., 2016d) results in both finite and unique completability, and therefore the number of samples required by the unfolding approach given in Remark 15 is for both finite and unique completability.

Lemma 29 *Assume that $d > 2$, $n > \max\{200, (r+2)(d-2)\}$ and $r \leq \frac{n}{6}$. Moreover, assume that the sampling probability satisfies*

$$p > \frac{1}{n^{d-2}} \max \left\{ 27 \, \log \left(\frac{2n}{\epsilon} \right) + 9 \, \log \left(\frac{8r(d-2)}{\epsilon} \right) + 18, 6r \right\} + \frac{1}{\sqrt[4]{n^{d-2}}} \tag{37}$$

Then, with probability at least $(1 - \epsilon) \left(1 - \exp(-\frac{\sqrt{n^{d-2}}}{2}) \right)^{n^2}$, $\mathcal{U}$ is finitely completable.

Proof The proof is similar to the proof of Lemma 24. ■

Remark 30 *Theorem 27 ensures that there exist enough number of algebraically independent polynomials and according to Theorem 26, these algebraically independent polynomials can be distinguished among all the polynomials, which has a unique solution that leads to the unique completion of the sampled tensor. Hence, Theorem 27 characterizes an information-theoretic sampling rate for completing the sampled data. However, one direction for future research is to propose an algorithm that solves this particular system of polynomials obtained from Theorem 26 in order to achieve the bound in Theorem 27 and close the gap. Note that there are some fast methods to approximate the zero of the system polynomials including generalizations of Newton-Raphson method, which is independent from this work as in this work we find the information-theoretic rate of sampling to ensure the proposed combinatorial deterministic conditions hold.*

6. Numerical Comparisons

In order to show the advantage of our proposed CP approach over the unfolding approach, we compare the lower bound on the total number of samples that is required for finite completability using an example. Since the bound on the number of samples for finiteness and uniqueness are the same for the unfolding approach and they are almost the same for the CP approach, we only consider finiteness bounds for this example. In particular, we consider a 7-way tensor $\mathcal{U}$ ($d = 7$) such that each dimension size is $n = 10^3$. We also consider the CP rank r which varies from 1 to 150. Figure 1 plots the bounds given in (14) (unfolding approach) and in (29) (CP approach) for the corresponding rank value, where $\epsilon = 0.001$. It is seen that the number of samples required by the proposed CP approach is substantially lower than that is required by the unfolding approach.

Observe that the unfolding approach does not take advantage of the structural property of the underlying CP model properly, and therefore our proposed multi-way analysis on CP manifold significantly outperforms the simple two-way analysis.

7. Conclusions

This paper is concerned with the low CP rank tensor completion problem and aims to derive fundamental conditions on the sampling pattern for finite and unique completability of a sampled tensor given its CP rank. In order to do so, a novel algebraic geometry analysis on the CP manifold is proposed. In particular, each sampled entry can be treated as a polynomials in terms of the entries of the components of the CP decomposition. We have defined a geometric pattern which classifies all CP decompositions such that each class includes only one decomposition of any tensor. We have
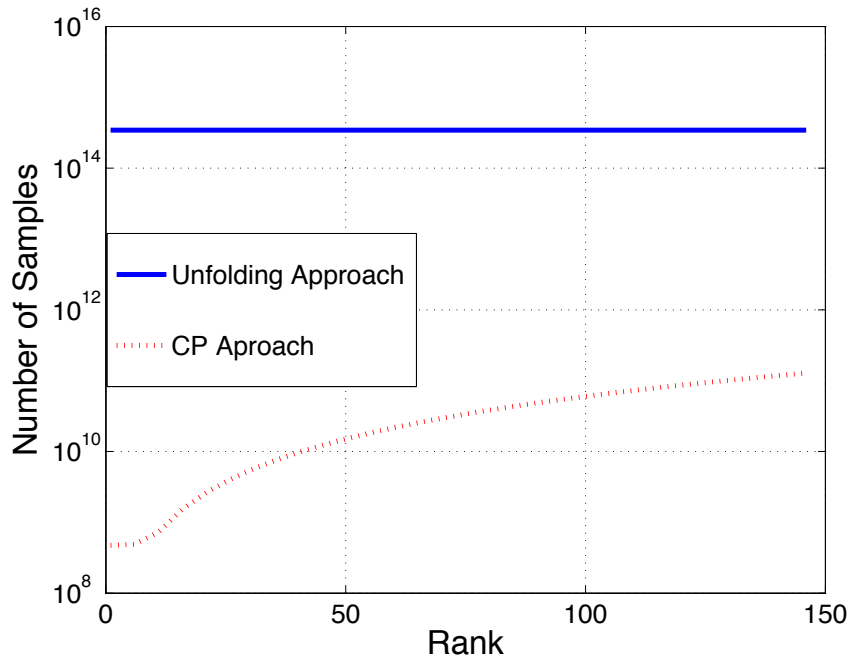
Figure 1: Lower bounds on the number of samples for a 7-way tensor.

shown that finite completability is equivalent to having a certain number of algebraically independent polynomials among all the defined polynomials based on the sampled entries. Furthermore, using the proposed classification, we can characterize the maximum number of algebraically independent polynomials in terms of a simple function of the sampling pattern. Moreover, we have developed several combinatorial tools that are used to bound the number of samples to ensure finite completability with high probability. Using these developed tools, we have treated three problems in this paper: (i) Characterizing the deterministic necessary and sufficient conditions on the sampling pattern, under which there are only finitely many completions given the CP rank, (ii) Characterizing deterministic sufficient conditions on the sampling pattern, under which there exists exactly one completion given the CP rank, (iii) Deriving lower bounds on the sampling probability or the number of samples such that the obtained deterministic conditions in Problems (i) and (ii) are satisfied with high probability. In addition, it is seen that our proposed CP analysis leads to an orders-of-magnitude lower number of samples than the unfolding approach that is based on analysis on the Grassmannian manifold.

Acknowledgments

References

Ralph Abraham, Jerrold E Marsden, and Tudor Ratiu. *Manifolds, Tensor Analysis, and Applications*, volume 75. Springer Science & Business Media, 2012.

Morteza Ashraphijuo and Xiaodong Wang. Characterization of deterministic and probabilistic sampling patterns for finite completability of low tensor-train rank tensor. *arXiv preprint:1703.07698*, 2017.

Morteza Ashraphijuo, Ramtin Madani, and Javad Lavaei. Inverse function theorem for polynomial equations using semidefinite programming. In *IEEE 54th Annual Conference on Decision and Control (CDC)*, pages 6589–6596, 2015.

Morteza Ashraphijuo, Vaneet Aggarwal, and Xiaodong Wang. Deterministic and probabilistic conditions for finite completability of low rank tensor. *arXiv preprint:1612.01597*, 2016a.

Morteza Ashraphijuo, Ramtin Madani, and Javad Lavaei. Characterization of rank-constrained feasibility problems via a finite number of convex programs. In *2016 IEEE 55th Conference on Decision and Control (CDC)*, pages 6544–6550, 2016b.

Morteza Ashraphijuo, Vaneet Aggarwal, and Xiaodong Wang. A characterization of sampling patterns for low-Tucker-rank tensor completion problem. In *IEEE International Symposium on Information Theory (ISIT)*, pages 531–535, 2017a.

Morteza Ashraphijuo, Xiaodong Wang, and Vaneet Aggarwal. A characterization of sampling patterns for low-rank multi-view data completion problem. In *IEEE International Symposium on Information Theory (ISIT)*, pages 1147–1151, 2017b.

Morteza Ashraphijuo, Xiaodong Wang, and Vaneet Aggarwal. Deterministic and probabilistic conditions for finite completability of low-rank multi-view data. *arXiv preprint:1701.00737*, 2017c.

Morteza Ashraphijuo, Xiaodong Wang, and Vaneet Aggarwal. Rank determination for low-rank data completion. *arXiv preprint:1707.00622*, 2017d.

Jian Feng Cai, Emmanuel J Candès, and Zuowei Shen. A singular value thresholding algorithm for matrix completion. *SIAM Journal on Optimization*, 20(4):1956–1982, 2010.

Emmanuel J Candès and Benjamin Recht. Exact matrix completion via convex optimization. *Foundations of Computational Mathematics*, 9(6):717–772, 2009.

Emmanuel J Candès and Terence Tao. The power of convex relaxation: Near-optimal matrix completion. *IEEE Transactions on Information Theory*, 56(5):2053–2080, 2010.

Emmanuel J Candès, Yonina C Eldar, Thomas Strohmer, and Vladislav Voroninski. Phase retrieval via matrix completion. *SIAM Journal on Imaging Sciences*, 6(1):199–225, 2013.

J Douglas Carroll and Jih-Jie Chang. Analysis of individual differences in multidimensional scaling via an N-way generalization of Eckart-Young decomposition. *Psychometrika*, 35(3):283–319, 1970.

Lieven De Lathauwer. A survey of tensor methods. In *IEEE International Symposium on Circuits and Systems*, pages 2773–2776, 2009.

Lieven De Lathauwer, Bart De Moor, and Joos Vandewalle. A multilinear singular value decomposition. *SIAM journal on Matrix Analysis and Applications*, 21(4):1253–1278, 2000.

Gregory Ely, Shuchin Aeron, Ning Hao, Misha E Kilmer, et al. 5d and 4d pre-stack seismic data completion using tensor nuclear norm (tnn). In *Society of Exploration Geophysicists*, 2013.

Silvia Gandy, Benjamin Recht, and Isao Yamada. Tensor completion and low-n-rank tensor recovery via convex optimization. *Inverse Problems*, 27(2):1–19, 2011.

Lars Grasedyck. Hierarchical singular value decomposition of tensors. *SIAM Journal on Matrix Analysis and Applications*, 31(4):2029–2054, 2010.

Lars Grasedyck, Daniel Kressner, and Christine Tobler. A literature survey of low-rank tensor approximation techniques. *GAMM-Mitteilungen*, 36(1):53–78, 2013.

Richard A Harshman. Foundations of the PARAFAC procedure: Models and conditions for an explanatory multi-modal factor analysis. *University of California at Los Angeles*, 1970.

Christopher J Hillar and Lek-Heng Lim. Most tensor problems are NP-hard. *Journal of the ACM (JACM)*, 60(6):45:1–45:39, 2013.

Sebastian Holtz, Thorsten Rohwedder, and Reinhold Schneider. On manifolds of tensors of fixed TT-rank. *Numerische Mathematik*, 120(4):701–731, 2012.

Prateek Jain and Sewoong Oh. Provable tensor factorization with missing data. In *Advances in Neural Information Processing Systems*, pages 1431–1439, 2014.

Misha E Kilmer, Karen Braman, Ning Hao, and Randy C Hoover. Third-order tensors as operators on matrices: A theoretical and computational framework with applications in imaging. *SIAM Journal on Matrix Analysis and Applications*, 34(1):148–172, 2013.

Tamara G Kolda. Orthogonal tensor decompositions. *SIAM Journal on Matrix Analysis and Applications*, 23(1):243–255, 2001.

Tamara G Kolda and Brett W Bader. Tensor decompositions and applications. *SIAM Review*, 51(3): 455–500, 2009.

Nadia Kreimer, Aaron Stanton, and Mauricio D Sacchi. Tensor completion based on nuclear norm minimization for 5D seismic data reconstruction. *Geophysics*, 78(6):V273–V284, 2013.

Daniel Kressner, Michael Steinlechner, and Bart Vandereycken. Low-rank tensor completion by Riemannian optimization. *BIT Numerical Mathematics*, 54(2):447–468, 2014.

Akshay Krishnamurthy and Aarti Singh. Low-rank matrix and tensor completion via adaptive sampling. In *Advances in Neural Information Processing Systems*, pages 836–844, 2013.

Lek-Heng Lim and Pierre Comon. Multiarray signal processing: Tensor decomposition meets compressed sensing. *Comptes Rendus Mecanique*, 338(6):311–320, 2010.

Ji Liu, Przemyslaw Musialski, Peter Wonka, and Jieping Ye. Tensor completion for estimating missing values in visual data. *IEEE Transactions on Pattern Analysis and Machine Intelligence*, 35(1):208–220, 2013.

Xiao-Yang Liu, Shuchin Aeron, Vaneet Aggarwal, and Xiaodong Wang. Low-tubal-rank tensor completion using alternating minimization. *arXiv preprint:1610.01690*, 2016.

Damien Muti and Salah Bourennane. Survey on tensor signal algebraic filtering. *Signal Processing*, 87(2):237–249, 2007.

Ivan V Oseledets. Tensor-train decomposition. *SIAM Journal on Scientific Computing*, 33(5):2295–2317, 2011.

Evangelos E Papalexakis, Christos Faloutsos, and Nicholas D Sidiropoulos. ParCube: Sparse parallelizable tensor decompositions. In *Joint European Conference on Machine Learning and Knowledge Discovery in Databases*, pages 521–536, 2012.

D Pimentel-Alarcón, Nigel Boston, and Robert D Nowak. Deterministic conditions for subspace identifiability from incomplete sampling. In *IEEE International Symposium on Information Theory (ISIT)*, pages 2191–2195, 2015.

D Pimentel-Alarcón, L Balzano, R Marcia, R Nowak, and R Willett. Group-sparse subspace clustering with missing data. In *IEEE Workshop on Statistical Signal Processing (SSP)*, pages 1–5, 2016a.

D Pimentel-Alarcón, EDU Robert D Nowak, and WISC EDU. The information-theoretic requirements of subspace clustering with missing data. In *International Conference on Machine Learning*, 2016b.

Daniel Pimentel-Alarcón, Laura Balzano, and Robert Nowak. Necessary and sufficient conditions for sketched subspace clustering. In *Allerton Conference on Communication, Control, and Computing*, 2016c.

Daniel Pimentel-Alarcón, Nigel Boston, and Robert Nowak. A characterization of deterministic sampling patterns for low-rank matrix completion. *IEEE Journal of Selected Topics in Signal Processing*, 10(4):623–636, 2016d.

Bernardino Romera-Paredes and Massimiliano Pontil. A new convex relaxation for tensor completion. In *Advances in Neural Information Processing Systems*, pages 2967–2975, 2013.

Nicholas D Sidiropoulos and Anastasios Kyrillidis. Multi-way compressed sensing for sparse low-rank tensors. *IEEE Signal Processing Letters*, 19(11):757–760, 2012.

Marco Signoretto, Raf Van de Plas, Bart De Moor, and Johan AK Suykens. Tensor versus matrix completion: a comparison with application to spectral data. *IEEE Signal Processing Letters*, 18(7):403–406, 2011.

Marco Signoretto, Quoc Tran Dinh, Lieven De Lathauwer, and Johan AK Suykens. Learning with tensors: a framework based on convex optimization and spectral regularization. *Machine Learning*, 94(3):303–351, 2014.

Alwin Stegeman and Nicholas D Sidiropoulos. On Kruskals uniqueness condition for the CANDECOMP/PARAFAC decomposition. *Linear Algebra and its Applications*, 420(2):540–552, 2007.

Bernd Sturmfels. *Solving Systems of Polynomial Equations*. Number 97. American Mathematical Society, 2002.

Jos MF ten Berge and Nikolaos D Sidiropoulos. On uniqueness in CANDECOMP/PARAFAC. *Psychometrika*, 67(3):399–409, 2002.

Ryota Tomioka, Kohei Hayashi, and Hisashi Kashima. Estimation of low-rank tensors via convex optimization. *arXiv preprint:1010.0789*, 2010.

Wenqi Wang, Vaneet Aggarwal, and Shuchin Aeron. Tensor completion by alternating minimization under the tensor train (TT) model. *arXiv preprint:1609.05587*, Sep. 2016.